# INTRODUCTION

TO

# TRIGONOMETRY.

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TO

# PLANE AND SPHERICAL

# TRIGONOMETRY,

BY


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THE Student is recommended, in the first instance, to make himself acquainted with Chapters I, V, and VI. He may then advance as far as page 89, and exercise himself in the Miscellaneous Problems, pages 114 to 124. After reading Chapter IV, Arts. 1 to 10, he may proceed with the Rules for the Solution of Spherical Triangles, pages 90 to 113; and the Miscellaneous Problems, pages 125 to 131; when he can turn his attention to the remaining portions of the book.

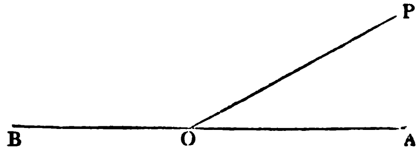
# PLANE TRIGONOMETRY.

## CHAPTER I.

1. Plane Trigonometry is that science which treats of the measurement of plane angles and triangles. As Geometry enables us to construct a triangle from three independent data, so, Trigonometry enables us, from the same data, expressed in numbers, to calculate its sides and angles.

2. In Geometry, an angle is defined to be the inclination of one straight line to another, and, therefore, can never exceed two right angles. But, in Trigonometry, there is no such restriction.

For, let B O A be a fixed line, and O P a line which revolves about O, and which at first coincided with O A. Then, when O P is in the position represented in the figure, it is said to have described the angle A O P. But this mode of conceiving an angle admits of extension to angles of any magnitude; for we may suppose O P to revolve beyond O B, and so to describe an angle



greater than two right angles, or, indeed, an angle of any magnitude whatever.

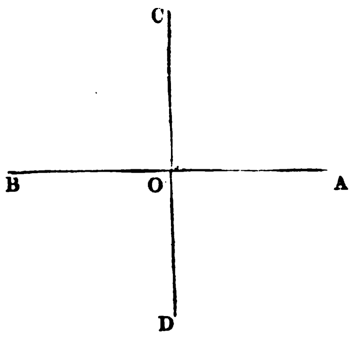
## On the Mode of Measuring Angles.

3. The circumference of a circle being divided into 360 equal parts, the angle at the centre subtended by one of these parts is called a *degree.* The degree is subdivided into 60 equal parts, called *minutes;* and the minute in 60 equal parts, called *seconds.* Degrees, minutes, and seconds are thus expressed— $^\circ\ '\ ''$; and when an angle is said to be $20^\circ\ 30'\ 40''$, we mean that it contains 20 degrees, 30 minutes, 40 seconds.

4. A right angle is the angle at the centre which is subtended by a *quadrant,* or *fourth* part of the circumference of a circle, and, therefore, contains 90 degrees.

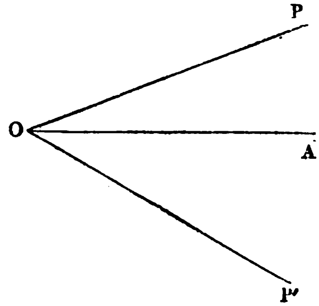
An angle is frequently denoted by a single letter. Thus, in fig. 4, the angle C P N would be called P; and P C N, C, &c.

## On the Use of the Signs + and −.




5. Let A O B, C O D be two lines at right angles to each other, then those lines drawn parallel to B O A are *positive,* if to the *right* of C D, *negative,* if to the *left:* and lines drawn parallel to C O D are *positive,* if *above, negative,* if *below,* A O B.

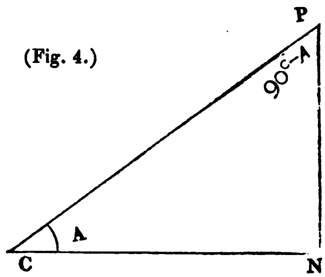
6. Again, suppose the line O P, by its revolution about O *upwards* from O A, to describe the angle A O P, and let the angles described in this manner be considered *positive*. Then if the line revolve *downwards* from O A, to describe the angle A O P′, this angle will properly be accounted *negative*.




## The Trigonometrical Ratios.

7. Let P C N be a triangle, right-angled at N, and let P C N contain any number of degrees, &c., which we may denote by A.

(Fig. 4.)




Then—

| | | | | |
|---|---|---|---|---|
| $\frac{PN}{CP}$ | or | $\frac{perp.}{hypoth.}$ | is called | Sine A. |
| $\frac{CN}{CP}$ | or | $\frac{base}{hyp.}$ | ,, | Cosine A. |
| $\frac{PN}{CN}$ | or | $\frac{perp.}{base}$ | ,, | Tangent A. |
| $\frac{CN}{PN}$ | or | $\frac{base}{perp.}$ | ,, | Cotangent A. |
| $\frac{CP}{CN}$ | or | $\frac{hyp.}{base}$ | ,, | Secant A. |
| $\frac{CP}{PN}$ | or | $\frac{hyp.}{perp.}$ | ,, | Cosecant A. |

and $1 -$ Cos. A is called Versine A.

For the sake of brevity the above are usually written— Sin. A, Cos. A, Tan. A, Cot. A, Sec. A, Cosec. A, and Vers. A.

8. The *Complement* of an angle is its defect from $90^\circ$, and is written thus, $90^\circ - A$.

The *Supplement* of an angle is its defect from $180^\circ$, and is written thus, $180^\circ - A$.

From fig. 4, P being the complement* of C, it will be seen that the cosine, cotangent, and cosecant are, respectively, the sine, tangent, and secant of the complement; for,

$$\text{Cos. A} = \frac{CN}{CP}, \text{ and Sin. } (90^\circ - A) = \frac{CN}{CP},$$

$$\therefore \text{Cos. A} = \text{Sin. } (90^\circ - A).$$

$$\text{Similarly, Cotan. A} = \text{Tan. } (90^\circ - A)$$

$$\text{and Cosec. A} = \text{Sec. } (90^\circ - A).$$

## Examples.

*a.* Find the complements of $45^\circ$; $50^\circ$; $60^\circ\ 10'\ 30''$; and $70^\circ\ 18'\ 10''$.

**Ans.** $45^\circ$; $40^\circ$; $29^\circ\ 49'\ 30''$; and $19^\circ\ 41'\ 50''$.

*b.* Find the supplements of $120^\circ$; $130^\circ\ 10'\ 40''$; $50^\circ\ 16'\ 45''$; and $145^\circ\ 10'\ 12''$.

**Ans.** $60^\circ$; $49^\circ\ 49'\ 20''$; $129^\circ\ 43'\ 15''$; and $34^\circ\ 49'\ 48''$.

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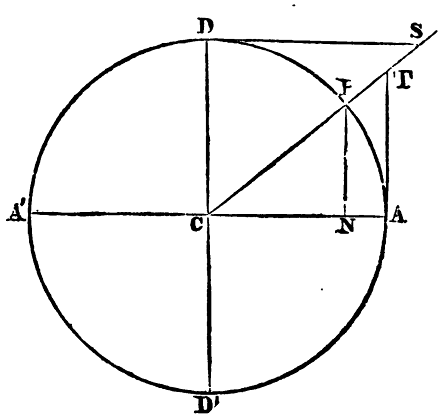
* For since N is a right angle, and the three interior angles of a triangle are equal to two right angles (*Euc.* Bk. I, p. 32), therefore $P + C = 90^\circ$, or $P = 90^\circ - C$, and $C = 90^\circ - P$, *i.e.*, P and C are the complements of each other.

9. The *meaning* of the names, sine, cosine, &c., will be seen more distinctly from another mode of defining them.

In the accompanying figure, let the diameters A A′, D D′, cut each other at right angles, and draw C P, making any angle A C P; also draw P N perpendicular to A C, and the tangents A T, D S, meeting C P, produced in T and S, then—




| | | | | | |
|---|---|---|---|---|---|
| The line | P N | is the | Sine | of angle | A C P. |
| „ | C N | „ | Cosine | „ | „ |
| „ | A T | „ | Tangent | „ | „ |
| „ | C T | „ | Secant | „ | „ |
| „ | D S | „ | Cotangent | „ | „ |
| „ | C S | „ | Cosecant | „ | „ |
| „ | A N | „ | Versine | „ | „ |

If we call the radius of the circle 1, then will A N = 1 − C N, or Vers. = 1 − Cos.

Hence we have some indication of the origin of the terms *sine*, *cosine*, &c. The word *sine* appears to be derived from the Latin word *sinus*, a bosom. Imagine the figure of an archer at P N. Then the *arc* A P may be supposed to represent his bow, and thus will get its name; its string, *i.e.*, the line joining A P, and which is also called the *chord* of the arc, would come against the *breast* (*sinus*) of the archer. The word *tangent* is derived from the Latin *tangere*, to touch,

because it touches the circle, as A T; and *secant*, from *secare*, to cut, because it cuts the circle, as C T. By *cosine* is meant the sine of the complement; and by *cotangent*, the tangent of the complement, &c.

10. The *Reciprocal* of a ratio is a fraction having 1 for its numerator, and the given ratio for its denominator: thus the reciprocal of the sine is $\frac{1}{\text{sine}}$, of the cosine, $\frac{1}{\text{cosine}}$, &c.

11. *To prove that the reciprocals of the sine, cosine and tangent of an angle are respectively equal to the cosecant, secant, and cotangent.*

By fig. 4—

$$\frac{1}{\text{Sin. A}} = \frac{1}{\frac{PN}{CP}} = \frac{CP}{PN} = \text{Cosec. A.} \qquad (1)$$

$$\frac{1}{\text{Cos. A}} = \frac{1}{\frac{CN}{CP}} = \frac{CP}{CN} = \text{Sec. A.} \qquad (2)$$

$$\frac{1}{\text{Tan. A}} = \frac{1}{\frac{PN}{CN}} = \frac{CN}{PN} = \text{Cot. A.} \qquad (3)$$

Conversely, we may prove that—

$$\frac{1}{\text{Cosec. A}} = \text{Sin. A.} \ (4). \qquad \frac{1}{\text{Sec. A}} = \text{Cos. A.} \ (5)$$

$$\frac{1}{\text{Cot. A}} = \text{Tan. A.} \ (6)$$

*Or, that the reciprocals of the cosecant, secant, and cotangent are respectively equal to the sine, cosine, and tangent.*

☞ The above formulæ and the two following, which are proved in a similar manner, should be carefully committed to memory:—

$$\text{Tan. A} = \frac{\text{Sin. A}}{\text{Cos. A}}. \ (7). \qquad \text{Cot. A} = \frac{\text{Cos. A}}{\text{Sin. A}}. \ (8)$$

## Examples.

*Simplify, by reducing to integral forms, the following expressions :—*

(1) $\frac{1}{\text{Cot. A}} \cdot \frac{1}{\text{Sec. A}}$.*

(2) $\frac{1}{\text{Sin. A}} \cdot \frac{1}{\text{Cos. A}}$.

(3) $\frac{1}{\text{Sec. A}} \cdot \text{Cot. A}$.

(4) $\frac{\text{Cos. A}}{\text{Cot. A}}$.

(5) $\frac{\text{Cos. B}}{\text{Sin.}^2\text{C} \cdot \text{Cos}^2\text{D}}$.

(6) $\frac{\text{Sec. A}}{1 - \text{Vers. A}}$.

(7) $\frac{\text{Sin.}^2\text{A}}{\text{Cosec.}^2\text{A}}$.

(8) $\frac{\text{Tan.A} \cdot \text{Cot.A} \cdot \text{Sec.A}}{1 - \text{Vers. A}}$.

(9) $\frac{1 - \text{Vers. A}}{\text{Sec. A}}$.

(10) $\frac{\text{Sin. A}}{(1 - \text{Vers. A}) \cdot \text{Cot.}^2\text{A}}$.

## Answers.

(1) Tan. A . Cos. A.

(2) Cosec. A . Sec. A.

(3) $\text{Cos.}^2\text{A}$ . Cosec. A

(4) Sin. A.

(5) Cos.B . $\text{Cosec.}^2\text{C}$ . $\text{Sec.}^2\text{D}$.

(6) $\text{Sec.}^2\text{A}$.

(7) $\text{Sin.}^4\text{A}$.

(8) $\text{Sec.}^2\text{A}$.

(9) $\text{Cos.}^2\text{A}$.

(10) $\text{Tan.}^3\text{A}$.

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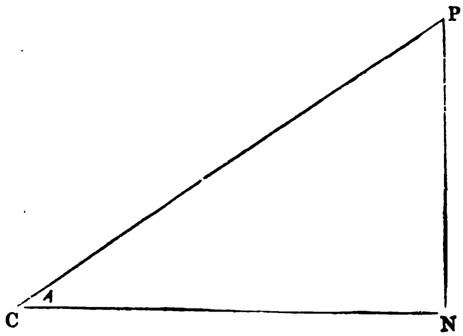
* In Example 1, we have $\frac{1}{\text{Cot.A}} = \text{Tan. A}$, by formula (6)

and $\frac{1}{\text{Sec.A}} = \text{Cos. A}$ „ (5)

$$\therefore \frac{1}{\text{Cot.A}} \cdot \frac{1}{\text{Sec.A}} = \text{Tan.A} \cdot \text{Cos.A}.$$

The other Examples may be simplified in the same manner.

12. *To prove that* $\text{Sin.}^2 A + \text{Cos.}^2 A = 1$.




In the right-angled triangle C P N, we have, by Euclid, Book I, Prop. 47:

$$PN^2 + CN^2 = CP^2.$$

Dividing this Equation by $CP^2$,

we have $\left(\frac{PN}{CP}\right)^2 + \left(\frac{CN}{CP}\right)^2 = \left(\frac{CP}{CP}\right)^2 = 1.$

or, $\quad \text{Sin.}^2 A + \text{Cos.}^2 A = 1. \qquad (9)$

From which formula, by transposing and taking the square root, we obtain

$\text{Sin. } A = \sqrt{1 - \text{Cos.}^2 A}$ (10), and $\text{Cos. } A = \sqrt{1 - \text{Sin.}^2 A}$. (11)

13. *To prove that*

$$\text{Sec.}^2 A = 1 + \text{Tan.}^2 A, \text{ and } \text{Cosec.}^2 A = 1 + \text{Cot.}^2 A.$$

From the fig. $\quad CP^2 = CN^2 + PN^2.$

Dividing by $CN^2$, $\left(\frac{CP}{CN}\right)^2 = \left(\frac{CN}{CN}\right)^2 + \left(\frac{PN}{CN}\right)^2$

or, $\quad \text{Sec.}^2 A = 1 + \text{Tan.}^2 A. \qquad (12)$

Again, since $\quad CP^2 = PN^2 + CN^2$

$$\therefore \left(\frac{CP}{PN}\right)^2 = \left(\frac{PN}{PN}\right)^2 + \left(\frac{CN}{PN}\right)^2.$$

or, $\quad \text{Cosec.}^2 A = 1 + \text{Cot.}^2 A. \qquad (13)$

14. *To express* Sin. A, *and* Cos. A, *in terms of* Tan. A.

By (4) $\text{Sin.A} = \frac{1}{\text{Cosec. A}}$, and by (13) $\text{Cosec. A} = \sqrt{1 + \text{Cot.}^2\text{A}}$

$$\therefore \text{Sin.A} = \frac{1}{\sqrt{1 + \text{Cot.}^2\text{A}}} = \frac{1}{\sqrt{1 + \frac{1}{\text{Tan.}^2\text{A}}}} = \frac{\text{Tan. A}}{\sqrt{\text{Tan.}^2\text{A} + 1}}$$

By (5) $\text{Cos. A} = \frac{1}{\text{Sec. A}}$, and by (12) $\text{Sec. A} = \sqrt{1 + \text{Tan.}^2\text{A}}$

$$\therefore \quad \text{Cos. A} = \frac{1}{\sqrt{1 + \text{Tan.}^2\text{A}}}.$$

15. *To express all the trigonometrical ratios in terms of the Sine.*

By (9) $\quad \text{Sin.}^2\text{A} + \text{Cos.}^2\text{A} = 1,$

$$\therefore \text{Cos.}^2\text{A} = 1 - \text{Sin.}^2\text{A},$$

$$\text{and Cos. A} = \pm\sqrt{1 - \text{Sin.}^2\text{A}}.$$

$$\text{Tan. A} = \frac{\text{Sin. A}}{\text{Cos. A}} = \pm\frac{\text{Sin. A}}{\sqrt{1 - \text{Sin.}^2\text{A}}}.$$

$$\text{Cot. A} = \frac{\text{Cos. A}}{\text{Sin. A}} = \pm\frac{\sqrt{1 - \text{Sin.}^2\text{A}}}{\text{Sin. A}}.$$

$$\text{Sec. A} = \frac{1}{\text{Cos. A}} = \pm\frac{1}{\sqrt{1 - \text{Sin.}^2\text{A}}}.$$

$$\text{Cosec. A} = \frac{1}{\text{Sin. A}}.$$

$$\text{Vers. A} = 1 - \text{Cos. A} = 1 \mp \sqrt{1 - \text{Sin.}^2\text{A}}.$$

*In a similar manner, express all the trigonometrical ratios in the terms of the Cosine.*

## Answers.

$\text{Sin. A} = \pm\sqrt{1 - \text{Cos.}^2\text{A}}.$ $\qquad$ $\text{Sec. A} = \frac{1}{\text{Cos. A}}.$

$\text{Tan. A} = \pm\frac{\sqrt{1 - \text{Cos.}^2\text{A}}}{\text{Cos. A}}.$ $\qquad$ $\text{Cosec. A} = \pm\frac{1}{\sqrt{1 - \text{Cos.}^2\text{A}}}.$

$\text{Cot. A} = \pm\frac{\text{Cos. A}}{\sqrt{1 - \text{Cos.}^2\text{A}}}.$ $\qquad$ $\text{Vers. A} = 1 - \text{Cos. A}.$

16. Given Sin. $x$ . Cos. A = Sin. A; find the value of Sin. $x$.

$$\text{Sin. } x \text{ . Cos. A} = \text{Sin. A.}$$

Dividing the equation by Cos. A,

$$\text{Sin. } x = \frac{\text{Sin. A}}{\text{Cos. A}} = \text{Tan. A.}$$

17. Given Sin. A . Tan. B = Sin. C . Cos. A; find Tan. B since Sin. A . Tan. B = Sin. C . Cos. A.

Dividing the equation by Sin. A, we have

$$\text{Tan. B} = \frac{\text{Sin. C . Cos. A}}{\text{Sin. A}} = \text{Sin. C . Cot. A.}$$

## Examples.

(i) Given Sin. $x$ . Cos. A = Tan. B . Sin. A; find Sin. $x$.
**Ans.** Sin. $x$ = Tan. B . Tan. A.

(ii) Given Cot. $x$ . Tan. $\theta$ = Sin. $\theta$ . Cos. $\phi$; find Cot. $x$.
**Ans.** Cot. $x$ = Cos. $\theta$ . Cos. $\phi$.

(iii) Given Sin. A . Tan. B = Cot. C . Tan. D; find Sin. A.
**Ans.** Sin. A = Cot. B . Cot. C . Tan. D.

(iv) Given Cos. $x$ . Sin. A = Cos. A; find Cos. $x$ and Sec. $x$.
**Ans.** Cos. $x$ = Cot. A and Sec. $x$ = Tan. A.

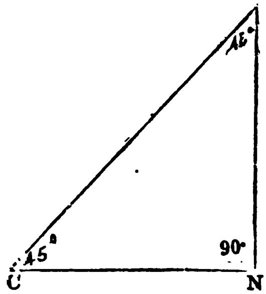
(v) Given Cos. $a$ = Cos. $b$ . Cos. $c$; find Cos. $b$.
**Ans.** Cos. $b$ = Cos. $a$ . Sec. $c$.

(vi) Given Cos. $a$ = Cot. B . Cot. C; find Cot. C.
**Ans.** Cot. C = Cos. $a$ . Tan. B.

(vii) Given Cos. B = Cot. $a$ . Tan. $c$; find Cot. $a$.
**Ans.** Cot. $a$ = Cos. B . Cot. $c$.

## On the Numerical Values of Angles.

18. *To find the values of* Sin. 45°; Cos. 45°, &c.




Let PCN be a right-angled isosceles triangle; then each of the angles P and C = 45°.

Since $CP^2 = PN^2 + CN^2$, and $PN = CN$

$\therefore\ CP^2 = 2PN^2$, or $CP = \sqrt{2}.PN$.

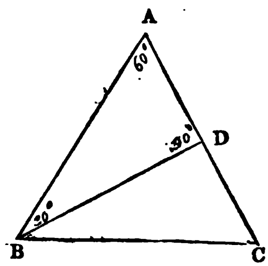
$$\text{Now, Sin. } 45^\circ = \frac{PN}{CP} = \frac{PN}{\sqrt{2}.PN} = \frac{1}{\sqrt{2}}.$$

$$\text{Tan. } 45^\circ = \frac{PN}{CN} = 1.$$

$$\text{Sec. } 45^\circ = \frac{CP}{PN} = \frac{\sqrt{2}.PN}{PN} = \sqrt{2}.$$

And Cos. 45° = Sin. 45° = $\frac{1}{\sqrt{2}}$; Cot. 45° = Tan. 45° = 1, &c.

19. *To find the values of* Sin. 30°; Cos. 60°; Tan. 30°; Cot. 60°; &c.




Let ABC be an equilateral triangle, and draw BD perpendicular to, and therefore bisecting AC. Then since A+B+C=180°, and these angles are equal, each is equal to 60°, and therefore ABD = 30°.

Now—

$$\text{Sin. } 30^\circ = \frac{AD}{AB} = \frac{\frac{1}{2}AB}{AB} = \tfrac{1}{2}.$$

$$\text{Cos. } 30^\circ = \sqrt{1 - \text{Sin.}^2 30} = \sqrt{1 - \tfrac{1}{4}} = \sqrt{\tfrac{3}{4}} = \frac{\sqrt{3}}{2} \text{ by (11)}$$

$$\text{Tan. } 30^\circ = \frac{\text{Sin. } 30^\circ}{\text{Cos. } 30^\circ} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}.$$

$$\text{Sec. } 30^\circ = \frac{1}{\text{Cos. } 30^\circ} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}.$$

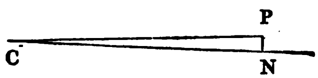
$$\text{Cot. } 30^\circ = \frac{\text{Cos. } 30^\circ}{\text{Sin. } 30^\circ} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}.$$

$$\text{Cosec. } 30^\circ = \frac{1}{\text{Sin. } 30^\circ} = \frac{1}{\frac{1}{2}} = 2.$$

And $\text{Sin. } 60^\circ = \text{Cos. } 30^\circ = \frac{\sqrt{3}}{2}$; $\text{Cos. } 60^\circ = \text{Sin. } 30^\circ = \frac{1}{2}$, &c.

20. *To find the values of* Sin. 0°; Cos. 0°; &c.

In triangle CPN, $\text{Sin. C} = \frac{PN}{CP}$.




But when $\angle PCN = 0^\circ$, PN vanishes, and CP coincides with, and therefore equals CN.

$$\text{Now, Sin. } 0^\circ = \frac{0}{CP} = 0.$$

$$\text{Cos. } 0^\circ = \frac{CN}{CP} = 1.$$

$$\text{Tan. } 0^\circ = \frac{0}{CN} = 0.$$

$$\text{Cot. } 0^\circ = \frac{CN}{0} = \infty^{ty} \text{ (infinity), \&c.}$$


21. *To find the values of* Sin. 90°; Cos. 90°; &c.

In triangle CPN, Sin. C $= \frac{PN}{CP}$.

But when C = 90°, PN = CP, and CN vanishes.

$$\therefore \text{ Sin. } 90° = \frac{PN}{CP} = 1.$$

$$\text{Cos. } 90° = \frac{0}{CP} = 0.$$

$$\text{Tan. } 90° = \frac{PN}{0} = \infty^{ty}, \text{ \&c.}$$

22. *To find the values of* Sin. 180°: Cos. 180°; &c.

In triangle CPN, Sin. C $= \frac{PN}{CP}$.

But when the angle = 180°,


PN vanishes, and CP = −CN. (Art. 5, p. 2.)

$$\text{Now,} \quad \text{Sin. } 180° = \frac{0}{CP} = 0.$$

$$\text{Cos. } 180° = -\frac{CN}{CP} = 1.$$

$$\text{Tan. } 180° = \frac{-PN}{0} = -\infty^{ty}.$$

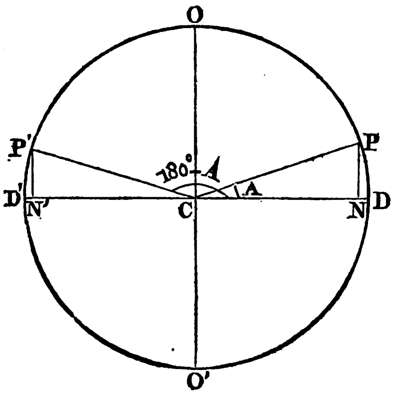
$$\text{\&c.} = \text{\&c.}$$

And so on for the other quadrants.

23. To prove that—

*The sine of an angle is equal to the sine of its supplement, and that*

*The cosine of an angle is equal to the cosine of its supplement, but is of different sign.*




Let DCP = A, and make D′CP′ = DCP, then DCP′ = (180° − D′CP′) = (180° − DCP) = (180° − A). Drop the perpendiculars P′N′, PN on DD′. Then the triangles PCN, P′CN′ are equal in all respects.

Now, $\text{Sin. } A = \dfrac{PN}{CP} = \dfrac{P'N'}{CP'}$ (because $PN = PN'$, and $CP = CP'$).

But $\text{Sin. } \overline{180^\circ - A} = \dfrac{P'N'}{CP'}$.

$\therefore \quad \text{Sin. } A = \text{Sin. } \overline{180^\circ - A}.$ (14)

Again, $\text{Cos. } A = \dfrac{CN}{CP} = \dfrac{CN'}{CP'}$.

and $\text{Cos. } \overline{180^\circ - A} = -\dfrac{CN'}{CP'}$, (Art. 5, p. 2.)

or, $-\text{Cos. } \overline{180^\circ - A} = \dfrac{CN'}{CP'}$,

$\therefore \quad \text{Cos. } A = -\text{Cos. } \overline{180^\circ - A}.$ (15)

Hence, also,

$$\text{Tan. A} = \frac{\text{Sin. A}}{\text{Cos. A}} = \frac{\text{Sin. } \overline{180^\circ - A}}{-\text{Cos. } \overline{180^\circ - A}} = -\text{Tan. } \overline{180^\circ - A}.$$

$$\text{Cot. A} = \frac{\text{Cos. A}}{\text{Sin. A}} = \frac{-\text{Cos. } \overline{180^\circ - A}}{\text{Sin. } \overline{180^\circ - A}} = -\text{Cot. } \overline{180^\circ - A}.$$

$$\text{Sec. A} = \frac{1}{\text{Cos. A}} = \frac{1}{-\text{Cos. } \overline{180^\circ - A}} = -\text{Sec. } \overline{180^\circ - A}.$$

$$\text{Cosec. A} = \frac{1}{\text{Sin. A}} = \frac{1}{\text{Sin. } \overline{180^\circ - A}} = \text{Cosec. } \overline{180^\circ - A}.$$

$$\text{Vers.A} = 1 - \text{Cos.A} = 1 - (-\text{Cos.}\overline{180^\circ - A}) = 1 + \text{Cos.}\overline{180^\circ - A}.$$

Hence when an angle is greater than 90°, and less than 180°, as D C P′ for instance, all its trigonometrical ratios, except the *sine, cosecant*, and *versine*, are *negative.*

24. *To determine the algebraic sign of a trigonometrical ratio in any equation when all the other terms are known.*

**Rule.**—Put the proper sign over each ratio; then those on one side of the equation must be made to produce the *same* result, *positive*, or *negative*, as those on the other side; as in the following example:—

Given, B = 100°, and C = 120°, to determine whether Cos. A is positive, or negative, in the equation

$$\overset{+}{\text{Cos. A}} = \overset{-}{\text{Tan. B}} \, . \, \overset{-}{\text{Cot. C}}.$$

Here, since B and C are both greater than 90°, their signs will be *negative*, by preceding Article, and the product of these two negative signs being +, the sign over Cos. A will also be +, or Cos. A will be *positive*, and A will therefore be less than 90°. If the sign over Cos. A had been −, A would have been greater than 90°, in which case the angle resulting from the above formula would have to be subtracted from 180°, since by (15)

$$-\text{Cos. A} = \text{Cos. } (180^\circ - \text{A}).$$

## Examples.

25. Determine whether Cos. $x$ is positive or negative in the equations:—

(1) Cos. $x$ = Cos. A . Sin. B . Tan. C,

(2) Cos. A = Cos. $x$ . Cot. B,

(3) Cotan. C = Cot. B . Cot. A . Cos, $x$,

(4) Tan. B . Cos. C = Tan. A . Cos. $x$,

when A = 50°, B = 100°, C = 120°.

**Ans.** (1) Negative. (3) Positive.
(2) Negative. (4) Positive.

26. Determine whether A is greater or less than 90°, in the following formulæ, the values of A, B and C being the same as before:—

(1) Tan. A = Tan. B . Cot. C.

(2) Cos. A = Tan. B . Cosec. C.

**Ans.** (1) Less. (2) Greater.

*Obs.*—When the value of an angle is to be determined from its *sine*, the above rule will not apply, since the sine is *positive*, whether the angle is greater or less than 90°. The ambiguity which thus arises can only be removed in particular cases.

---

## Exercise on the preceding Chapter.

1. What is meant by plane trigonometry?

2. In what respect does the trigonometrical definition of an angle differ from the geometrical?

3. Explain how angles are measured.

4. Explain the uses of the signs + and −.

5. Define the trigonometrical ratios.

6. Give definitions of the complement and supplement of an angle.

7. If Sin. A $= m$, express all the trigonometrical ratios in terms of $m$.

## Answers.

$$\text{Cos. A} = \pm\sqrt{1 - m^2}. \qquad \text{Sec. A} = \pm\frac{1}{\sqrt{1 - m^2}}.$$

$$\text{Tan. A} = \pm\frac{m}{\sqrt{1 - m^2}}. \qquad \text{Cosec. A} = \frac{1}{m}.$$

$$\text{Cot. A} = \pm\frac{\sqrt{1 - m^2}}{m}. \qquad \text{Vers. A} = 1 \mp \sqrt{1 - m^2}.$$

8. If Cos. A $= n$, express Sin. A, Tan. A, &c., in terms of $n$.

## Answers.

$$\text{Sin. A} = \pm\sqrt{1 - n^2}. \qquad \text{Cosec. A} = \pm\frac{1}{\sqrt{1 - n^2}}.$$

$$\text{Tan. A} = \pm\frac{\sqrt{1 - n^2}}{n}. \qquad \text{Cotan. A} = \pm\frac{n}{\sqrt{1 - n^2}}.$$

$$\text{Sec. A} = \frac{1}{n}. \qquad \text{Vers. A} = 1 - n.$$

9. Prove that

$$\begin{aligned} \text{Sin. A} &= \text{Cos. } \overline{90^\circ - A}; \\ \text{Tan. A} &= \text{Cot. } \overline{90^\circ - A}; \\ \text{Sec. A} &= \text{Cosec. } \overline{90^\circ - A}. \end{aligned}$$

10. If $3\ \text{Sin.}^2 A + \text{Cos.}^2 A = 2$, find the value of Sin. A.

Ans. $\dfrac{1}{\sqrt{2}}$.

11. If $2\ \text{Sin.}^2 A + \text{Cos.}^2 A = \frac{1}{2}$, find the value of Cos. A, and Sec. A.

Ans. $\text{Cos. A} = \sqrt{\dfrac{3}{2}}, \qquad \text{Sec. A} = \sqrt{\dfrac{2}{3}}.$

12. If Tan. A $= \frac{1}{2}$, find the value of Sin. A, and Cosec. A.

$$\text{Ans. Sin. A} = \frac{1}{\sqrt{5}}, \text{ Cosec. A} = \sqrt{5}.$$

13. If Tan. A $= \frac{1}{3}$, find the value of Cos. A.

$$\text{Ans. } \frac{3}{\sqrt{10}}.$$

14. Prove the following formulæ:—

$$\frac{\text{Sin.}^2\text{A}}{\text{Cos.}^2\text{A}} + 1 = \text{Sec.}^2\text{A}.$$

$$\frac{\text{Cos.}^2\text{A}}{\text{Sin.}^2\text{A}} + 1 = \text{Cosec.}^2\text{A}.$$

$$\text{Tan. A . Cot. A} = 1.*$$

$$\text{Cot.}^2\text{A} - \text{Cos.}^2\text{A} = \text{Cot.}^2\text{A . Cos.}^2\text{A}.$$

$$\text{Tan. A} + \text{Cot. A} = \text{Sec. A . Cosec. A}.$$

$$\text{Tan.}^2\text{A} - \text{Sin.}^2\text{A} = \text{Tan.}^2\text{A . Sin.}^2\text{A}.$$

$$\frac{\text{Cos. A . Sec. A}}{\text{Sin. A}} = \text{Cosec. A}.$$

$$\text{Cos. }\theta = \text{Cot.}^3\theta\text{ . Sin. }\theta\text{ . Tan.}^2\theta. \qquad \text{Vers. }\theta = \frac{\text{Sec. }\theta - 1}{\text{Sec. }\theta}.$$

15. If Tan. $\theta = \frac{a}{b}$, show that Sin. $\theta = \frac{a}{\sqrt{a^2 + b^2}}$, and Cos. $\theta = \frac{b}{\sqrt{a^2 + b^2}}$.

16. Show that

$$\text{Sin. }\theta\text{ . Cot. }\theta = 1, \quad \text{when } \theta = 0°,$$

and

$$\text{Sin. }\theta + \text{Cos. }\theta + \text{Tan. }\theta = \text{Sec. }\theta, \quad \text{when } \theta = 180°.$$

---

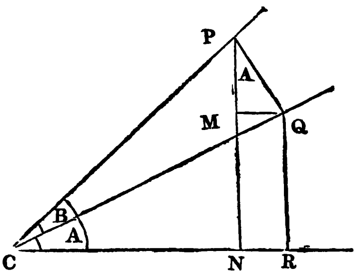
* In this and following examples in question 14, express all the ratios in terms of sine and cosine, reduce, and cancel.

## CHAPTER II.

### On Formulæ Involving Two Angles.

1. *To find the Sine and Cosine of the sum of two angles.*

Let QCR = A, PCQ = B.




From any point P in CP, draw PN, PQ, perpendiculars to CR, CQ; and QM, QR, perpendiculars to PN and CR.

It is evident that QPM = 90° − PQM = MQC = QCR = A.

Now, $\text{Sin.}\,(A + B) = \text{Sin.}\,PCN = \dfrac{PN}{CP} = \dfrac{MN + PM}{CP}$

$$= \frac{QR + PM}{CP} = \frac{QR}{CP} + \frac{PM}{CP}.$$

Multiply the first of these ratios by $\dfrac{CQ}{CQ}$, and the second by $\dfrac{PQ}{PQ}$.

Then, $\text{Sin.}\,(A + B) = \dfrac{QR}{CP}\cdot\dfrac{CQ}{CQ} + \dfrac{PM}{CP}\cdot\dfrac{PQ}{PQ}$

$$= \frac{QR}{CQ}\cdot\frac{CQ}{CP} + \frac{PM}{PQ}\cdot\frac{PQ}{CP}$$

But $\qquad \dfrac{QR}{CQ} = \text{Sin.}\,A,\ \dfrac{CQ}{CP} = \text{Cos.}\,B;$

$\dfrac{PM}{PQ} = \text{Cos.}\,QPM,$ or $\text{Cos.}\,A$; and $\dfrac{PQ}{CP} = \text{Sin.}\,B.$

$$\therefore\ \text{Sin.}\,(A + B) = \text{Sin.}\,A\,.\,\text{Cos.}\,B + \text{Cos.}\,A\,.\,\text{Sin.}\,B. \qquad (16)$$

Again, $\text{Cos.}(A+B) = \frac{CN}{CP} = \frac{CR - NR}{CP} = \frac{CR - QM}{CP}$

$$= \frac{CR}{CP} - \frac{QM}{CP}$$

$$= \frac{CR}{CP} \cdot \frac{CQ}{CQ} - \frac{QM}{CP} \cdot \frac{PQ}{PQ}$$

$$= \frac{CR}{CQ} \cdot \frac{CQ}{CP} - \frac{QM}{PQ} \cdot \frac{PQ}{CP}$$

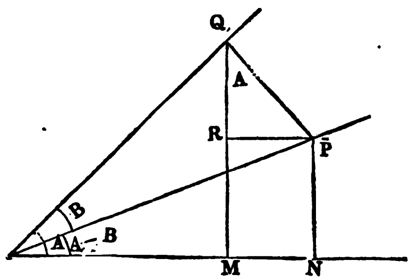
$$= \text{Cos.A}\,.\,\text{Cos.B} - \text{Sin.A}\,.\,\text{Sin.B}. \quad (17)$$

2 *To find the Sine and Cosine of the difference of two angles.*

Let $QCN = A$, $PCQ = B$.

Then $PCN = A - B$.

Take any point P in CP, and drop perpendiculars PQ, PN on CQ and CN; and also drop the perpendicular QM on CN, and PR on QM.




It is evident that $PQR = 90° - CQM = QCM = A$.

Now, $\text{Sin.}(A-B) = \text{Sin.PCN} = \frac{PN}{CP} = \frac{QM - QR}{CP} = \frac{QM}{CP} - \frac{QR}{CP}$

$$= \frac{QM}{CP} \cdot \frac{CQ}{CQ} - \frac{QR}{CP} \cdot \frac{PQ}{PQ}$$

$$= \frac{QM}{CQ} \cdot \frac{CQ}{CP} - \frac{QR}{PQ} \cdot \frac{PQ}{CP}$$

$$= \text{Sin. A}\,.\,\text{Cos. B} - \text{Cos. A}\,.\,\text{Sin. B}. \quad (18)$$

$$\text{Again, Cos. }(A-B) = \frac{CN}{CP} = \frac{CM + MN}{CP} = \frac{CM + PR}{CP}$$

$$= \frac{CM}{CP} + \frac{PR}{CP}$$

$$= \frac{CM}{CQ} \cdot \frac{CQ}{CP} + \frac{PR}{PQ} \cdot \frac{PQ}{CP}$$

$$= \text{Cos.A . Cos.B + Sin. A . Sin.B.} \quad (19)$$

3. ***Formulæ immediately deduced from the four preceding:—***

By 16, $\text{Sin. }(A + B) = \text{Sin. A . Cos. B + Cos. A . Sin. B.}$

By 18, $\text{Sin. }(A - B) = \text{Sin. A . Cos. B − Cos. A . Sin. B.}$

∴ By addition,

$$\text{Sin. }(A + B) + \text{Sin. }(A-B) = 2 \text{ Sin. A . Cos. B.} \quad (20)$$

And by subtraction,

$$\text{Sin. }(A + B) - \text{Sin. }(A - B) = 2 \text{ Cos. A . Sin. B.} \quad (21)$$

By (17), $\text{Cos. }(A + B) = \text{Cos. A . Cos. B − Sin. A . Sin. B.}$

By (19), $\text{Cos. }(A - B) = \text{Cos. A . Cos. B + Sin. A . Sin. B.}$

∴ By addition,

$$\text{Cos. }(A + B) + \text{Cos. }(A - B) = 2 \text{ Cos. A . Cos. B.} \quad (22)$$

And by subtraction,

$$\text{Cos. }(A+B) - \text{Cos. }(A-B) = -2 \text{ Sin. A . Sin. B.} \quad (23)$$

$$\text{Since } A = \frac{A+B}{2} + \frac{A-B}{2}$$

$$\text{And } B = \frac{A+B}{2} - \frac{A-B}{2}.$$

$$\therefore \text{ Sin. A} = \text{Sin.}\left(\frac{A+B}{2} + \frac{A-B}{2}\right)$$

$$= \text{Sin.}\frac{A+B}{2} \cdot \text{Cos.}\frac{A-B}{2} + \text{Cos.}\frac{A+B}{2} \cdot \text{Sin.}\frac{A-B}{2} \text{ by (16)}$$

$$\text{and Sin. B} = \text{Sin.}\left(\frac{A+B}{2} - \frac{A-B}{2}\right)$$

$$= \text{Sin.}\frac{A+B}{2} \,.\, \text{Cos.}\frac{A-B}{2} - \text{Cos.}\frac{A+B}{2} \,.\, \text{Sin.}\frac{A-B}{2}. \text{ by}(18)$$

$$\therefore \text{ Sin. A} + \text{Sin. B} = 2 \text{ Sin.}\frac{A+B}{2} \,.\, \text{Cos.}\frac{A-B}{2}, \qquad (24)$$

$$\text{and Sin. A} - \text{Sin. B} = 2 \text{ Cos.}\frac{A+B}{2} \text{ Sin.}\frac{A-B}{2}. \qquad (25)$$

Again—

$$\text{Cos. A} = \text{Cos.}\left(\frac{A+B}{2} + \frac{A-B}{2}\right)$$

$$= \text{Cos.}\frac{A+B}{2} \,.\, \text{Cos.}\frac{A-B}{2} - \text{Sin.}\frac{A+B}{2} \,.\, \text{Sin.}\frac{A-B}{2}. \text{ by } (17)$$

$$\text{Cos. B} = \text{Cos.}\left(\frac{A+B}{2} - \frac{A-B}{2}\right)$$

$$= \text{Cos.}\frac{A+B}{2} \,.\, \text{Cos.}\frac{A-B}{2} + \text{Sin.}\frac{A+B}{2} \,.\, \text{Sin.}\frac{A-B}{2}. \text{ by } (19)$$

$$\therefore \text{ Cos. A} + \text{Cos. B} = 2 \text{ Cos.}\frac{A+B}{2} \text{ Cos.}\frac{A-B}{2} \qquad (26)$$

$$\text{and Cos. A} - \text{Cos. B} = -2 \text{ Sin.}\frac{A+B}{2} \,.\, \text{Sin.}\frac{A-B}{2}. \qquad (27)$$

4. *To find the values of* Sin. 2 A, *and* Cos. 2 A, *in terms of* Sin. A, *and* Cos. A.

By (16), Sin. (A+B) = Sin. A . Cos. B + Cos. A . Sin. B,

and by (17), Cos. (A+B) = Cos. A . Cos. B − Sin. A . Sin. B.

These formulæ being true for *any* values of B and A, are true when B is equal to A, in which case we shall have

$$\text{Sin.}\ (A + A) = \text{Sin.}\ A\,.\,\text{Cos.}\ A + \text{Cos.}\ A\,.\,\text{Sin.}\ A,$$

$$\text{Cos.}\ (A + A) = \text{Cos.}\ A\,.\,\text{Cos.}\ A - \text{Sin.}\ A\,.\,\text{Sin.}\ A.$$

Or, $$\text{Sin.}\ 2A = 2\ \text{Sin.}\ A\,.\,\text{Cos.}\ A, \qquad (28)$$

$$\text{Cos.}\ 2A = \text{Cos.}^2 A - \text{Sin.}^2 A. \qquad (29)$$

5. If we put $\frac{A}{2}$ for A in (28) and (29), we shall have

$$\text{Sin.}\ A = 2\ \text{Sin.}\frac{A}{2}\,.\,\text{Cos.}\frac{A}{2}.$$

$$\text{Cos.}\ A = \text{Cos.}^2\frac{A}{2} - \text{Sin.}^2\frac{A}{2}.$$

Similarly, from (9), $1 = \text{Cos.}^2\frac{A}{2} + \text{Sin.}^2\frac{A}{2}.$

$\therefore$ By addition, $1 + \text{Cos.}\ A = 2\ \text{Cos.}^2\frac{A}{2}. \qquad (30)$

And by subtraction, $1 - \text{Cos.}\ A = 2\ \text{Sin.}^2\frac{A}{2}. \qquad (31)$

From (31), $2\ \text{Sin.}^2\frac{A}{2} = 1 - \text{Cos.}\ A = \text{Vers.}\ A$

$$\therefore\quad \text{Sin.}^2\frac{A}{2} = \tfrac{1}{2}\text{Vers.}\ A.^*$$

$$\text{Or, Sin.}\ \frac{A}{2} = \sqrt{\text{Hav.}\ A}. \qquad (32)$$

---

* $\frac{1}{2}$ Vers. A is called Haversine A, and generally written thus, Hav. A.

6. *To prove that*

$$\frac{\text{Sin. A} + \text{Sin. B}}{\text{Sin. A} - \text{Sin. B}} = \frac{\text{Tan. } \frac{1}{2} (\text{A} + \text{B})}{\text{Tan. } \frac{1}{2} (\text{A} - \text{B})}$$

$$\frac{\text{Sin. A} + \text{Sin. B}}{\text{Sin. A} - \text{Sin. B}} = \frac{2 \text{ Sin. } \frac{1}{2} (\text{A} + \text{B}) \text{ Cos. } \frac{1}{2} (\text{A} - \text{B})}{2 \text{ Cos. } \frac{1}{2} (\text{A} + \text{B}) \text{ Sin. } \frac{1}{2} (\text{A} - \text{B})} \quad \begin{matrix}(24)\\(25)\end{matrix}$$

$$= \text{Tan. } \tfrac{1}{2} (\text{A} + \text{B}) \,.\, \text{Cot. } \tfrac{1}{2} (\text{A} - \text{B})$$

$$= \text{Tan. } \tfrac{1}{2} (\text{A} + \text{B}) \,.\, \frac{1}{\text{Tan. } \frac{1}{2} (\text{A} - \text{B})} \quad (3)$$

$$= \frac{\text{Tan. } \frac{1}{2} (\text{A} + \text{B})}{\text{Tan. } \frac{1}{2} (\text{A} - \text{B})}. \quad (33)$$

7. *To find the tangent of the sum and difference of two angles.*

$$\text{Tan.}(\text{A}+\text{B}) = \frac{\text{Sin.}(\text{A}+\text{B})}{\text{Cos.}(\text{A}+\text{B})} = \frac{\text{Sin.A.Cos.B} + \text{Cos.A.Sin.B.}}{\text{Cos.A.Cos.B} - \text{Sin.A.Sin.B.}} \begin{matrix}(16)\\(17)\end{matrix}$$

Dividing each term by Cos. A . Cos. B, we have

$$\text{Tan. } (\text{A} + \text{B}) = \frac{\dfrac{\text{Sin. A . Cos. B}}{\text{Cos. A . Cos. B}} + \dfrac{\text{Cos. A . Sin. B}}{\text{Cos. A . Cos. B}}}{\dfrac{\text{Cos. A . Cos. B}}{\text{Cos. A . Cos. B}} - \dfrac{\text{Sin. A . Sin. B}}{\text{Cos. A . Cos. B}}} =$$

$$\frac{\text{Tan. A} + \text{Tan. B}}{1 - \text{Tan. A . Tan. B}}. \quad (34)$$

Similarly, $\text{Tan. } (\text{A} - \text{B}) = \dfrac{\text{Tan. A} - \text{Tan. B}}{1 + \text{Tan. A . Tan. B}}$. (35)

Putting B = A, in (34), we have

$$\text{Tan. } (\text{A} + \text{A}) = \frac{\text{Tan. A} + \text{Tan. A}}{1 - \text{Tan. A . Tan. A}}$$

$$\text{or, } \quad \text{Tan. } 2 \text{ A} = \frac{2 \text{ Tan. A}}{1 - \text{Tan.}^2\text{A}}. \quad (36)$$

8. *To prove that*

Sin. $(90^\circ + A) = \text{Cos.} A (37)$; and Cos. $(90^\circ + A) = -\text{Sin.} A. (38)$

By (16) Sin. $(90^\circ + A) = \text{Sin. } 90^\circ. \text{ Cos. } A + \text{Cos. } 90^\circ. \text{ Sin. } A$

By Sin. $90^\circ = 1$, and Cos. $90^\circ = 0$. (Art. 20 and 21, Ch. I)

$$\therefore \text{Sin. } (90^\circ + A) = 1 \times \text{Cos. } A + 0 \times \text{Sin. } A.$$

$$\therefore \text{Sin. } (90^\circ + A) = \text{Cos. } A. \qquad (39)$$

Again, by (17), Cos. $(90^\circ + A) = \text{Cos.} 90^\circ. \text{Cos.} A - \text{Sin.} 90^\circ. \text{Sin.} A.$

$$= 0 \times \text{Cos. } A - 1 \times \text{Sin. } A.$$

$$= -\text{Sin. } A. \qquad (40)$$

9. To find the values of Sin. $75^\circ$, and Cos. $75^\circ$.

$$\text{Sin. } 75^\circ = \text{Sin. } (45^\circ + 30^\circ) =$$

$$\text{Sin. } 45^\circ . \text{ Cos. } 30^\circ + \text{Cos. } 45^\circ . \text{ Sin. } 30^\circ. \qquad \text{by (16)}$$

But Sin. $45^\circ = \text{Cos.} 45^\circ = \frac{1}{2}\sqrt{2}$, Sin. $30^\circ = \frac{1}{2}$, Cos. $30^\circ = \frac{1}{2}\sqrt{3}$

(Arts. 18 and 19, Ch. I).

$$\therefore \text{Sin. } 75^\circ = \tfrac{1}{2}\sqrt{2} . \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{2} . \tfrac{1}{2} = \frac{\sqrt{3} + 1}{\sqrt{8}}.$$

$$\text{Cos.} 75^\circ = \text{Cos.} (45^\circ + 30^\circ) = \text{Cos.} 45^\circ. \text{Cos.} 30^\circ - \text{Sin.} 45^\circ. \text{Sin.} 30^\circ.$$

$$= \tfrac{1}{2}\sqrt{2} . \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{2} . \tfrac{1}{2}.$$

$$= \frac{\sqrt{3} - 1}{\sqrt{8}}.$$

10. *Find the values of* Sin. $15^\circ$; Sin. $105^\circ$; Cos. $135^\circ$; Cos. $150^\circ$.

## Answers.

Sin. $15^\circ = \dfrac{\sqrt{3} - 1}{\sqrt{8}}$. Cos. $135^\circ = \dfrac{-\sqrt{2}}{2}$.

Sin. $105^\circ = \dfrac{\sqrt{3} + 1}{\sqrt{8}}$. Cos. $150^\circ = \dfrac{-\sqrt{3}}{2}$.

## Examples for Practice.

1. Prove that $\text{Sin.}\ (\alpha + \beta)\ .\ \text{Sin.}\ (\alpha - \beta) = \text{Sin.}^2\alpha - \text{Sin.}^2\beta$.

2. $\text{Sin.}\ (\alpha + \beta)\ .\ \text{Sin.}\ (\alpha - \beta) = \text{Cos.}^2\beta - \text{Cos.}^2\alpha$.

3. $\text{Cos.}\ (\theta + \phi)\ .\ \text{Cos.}\ (\theta - \phi) = \text{Cos.}^2\theta - \text{Cos.}^2\phi$.

4. $\dfrac{\text{Sin.}\ (\alpha + \beta)}{\text{Cos.}\ \alpha\ .\ \text{Cos.}\ \beta} = \text{Tan.}\ \alpha + \text{Tan.}\ \beta$.

5. $\dfrac{\text{Sin.}\ (\alpha + \beta)}{\text{Sin.}\ \alpha\ .\ \text{Sin.}\ \beta} = \text{Cot.}\ \alpha + \text{Cot.}\ \beta$.

6. $\text{Tan.}\ \theta = \text{Cot.}\ \theta - 2\ \text{Cot.}\ 2\ \theta$.

7. $\text{Cosec.}\ 2\ \theta + \text{Cot.}\ 2\ \theta = \text{Cot.}\ \theta$.

8. $\text{Cot.}\ \theta + \text{Tan.}\ \theta = 2\ \text{Cosec.}\ 2\ \theta$.

9. $\text{Cot.}\ \theta - \text{Tan.}\ \theta = 2\ \text{Cot.}\ 2\ \theta$.

10. $2\ \text{Cosec.}\ 2\ \theta = \text{Sec.}\ \theta\ .\ \text{Cosec.}\ \theta$.

11. $\text{Cos.}\ \theta = \dfrac{1 - \text{Tan.}^2\ \dfrac{\theta}{2}}{1 + \text{Tan.}^2\ \dfrac{\theta}{2}}$.

12. $\dfrac{\text{Cos.}\ \theta + \text{Sin.}\ \theta}{\text{Cos.}\ \theta - \text{Sin.}\ \theta} = \text{Tan.}\ 2\ \theta + \text{Sec.}\ 2\ \theta$.

13. $\text{Sin.}\ 3\ A = 3\ \text{Sin.}\ A - 4\ \text{Sin.}^3 A$.

14. $\text{Tan.}\ (45^\circ + A) = \dfrac{1 + \text{Tan.}\ A}{1 - \text{Tan.}\ A}$.

15. $\text{Tan.}\ (45^\circ - A) = \dfrac{1 - \text{Tan.}\ A}{1 + \text{Tan.}\ A}$.

16. If $\text{Tan.}\ \theta = m$, and $\text{Tan.}\ \phi = n$, prove that

$$\text{Tan.}\ (\theta + \phi) = \frac{m + n}{1 - mn}; \qquad \text{Tan.}\ (\theta - \phi) = \frac{m - n}{1 + mn}.$$

## CHAPTER III.

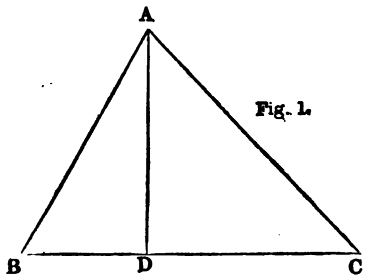
### On the Solution of Triangles.

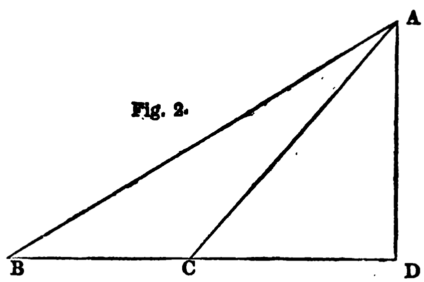
1. A triangle consists of six parts, viz.: three sides, and three angles. Any three of these being given (provided that one is a *side*) the others may be found.

2. The angles of the triangle are generally denoted by A, B, C, and the sides respectively opposite to them by *a, b, c*.

PROPOSITION I.

*The sides of a triangle are proportional to the sines of the opposite angles.*




Let A B C be any triangle, and from A let fall the perpendicular A D on B C, or B C produced, figs. (1) and (2). Then in fig. (1), where angle C is *acute*, we have

$$\frac{AD}{c} = \text{Sin. B, or } AD = c \text{ Sin. B},$$

and $$\frac{AD}{b} = \text{Sin. C, or } AD = b \,.\, \text{Sin. C}.$$

$$\therefore\ c \,.\, \text{Sin. B} = b \,.\, \text{Sin. C}.$$

And in fig. (2), where angle C is *obtuse*, we have

$$\frac{AD}{c} = \text{Sin. B, or } AD = c \text{ . Sin. B,}$$

and $$\frac{AD}{b} = \text{Sin. ACD} = \text{Sin. ACB.} \qquad \text{by (14)}$$

Since ACB is the supplement of ACD.

$$\therefore \; AD = b \text{ . Sin. ACB, or } b \text{ . Sin. C.}$$

$\therefore$ In each case,

$$c \text{ . Sin. B} = b \text{ . Sin. C,}$$

$$\text{or, } \frac{b}{c} = \frac{\text{Sin. B}}{\text{Sin. C}},$$

$$\textit{i.e.,} \; b : c :: \text{Sin. B : Sin. C,} \qquad \text{(A)}$$

Which proves Rule II.

## Proposition II.

*The square of any side of a triangle is equal to the sum of the squares of the other two sides, minus twice the product of those two sides, and the cosine of the angle included by them.*

First, let the triangle A B C be acute angled, C being an acute angle. (See fig. 1, preceding page.)

Then by Euc. II, 13,

$$AB^2 = BC^2 + AC^2 - 2\,BC \text{ . } CD,$$

and $$CD = AC \text{ . Cos. C.}$$

$$\therefore \; AB^2 = BC^2 + AC^2 - 2BC \text{ . } AC \text{ . Cos. C,}$$

or $$c^2 = a^2 + b^2 - 2\,ab \text{ . Cos. C.}$$

Secondly, suppose C an obtuse angle (fig. 2).

Then by Euc. II, 12,

$$AB^2 = BC^2 + AC^2 + 2BC \text{ . } CD,$$

and $$CD = AC\,(\text{Cos. } 180^\circ - C) = -\,AC \text{ . Cos. C.} \qquad (15)$$

$$\therefore \; AB^2 = BC^2 + AC^2 - 2\,BC \text{ . } AC \text{ . Cos. C,}$$

$$\textit{i.e.,} \; c^2 = a^2 + b^2 - 2\,ab \text{ Cos. C.} \qquad \text{(B)}$$

*Cor. Hence we may express the cosine of an angle in terms of the sides.*

Since by (B), $c^2 = a^2 + b^2 - 2ab.\text{Cos. C}$

$\therefore\ 2ab.\text{Cos. C} = a^2 + b^2 - c^2$

or, $$\text{Cos. C} = \frac{a^2 + b^2 - c^2}{2\,ab}. \qquad \text{(C)}$$

## PROPOSITION III.

*To adapt the formula,* $\text{Cos.A} = \dfrac{b^2 + c^2 - a^2}{2\,bc}$ *to logarithmic computation.*

Since Vers. A $= 1 -$ Cos. A,

or, 2 Hav. A $= 1 -$ Cos. A,

$$\therefore\ 2\ \text{Hav. A} = 1 - \frac{b^2+c^2-a^2}{2\,bc}$$

$$= \frac{2\,bc-b^2-c^2+a^2}{2\,bc}$$

$$= \frac{a^2-b^2+2\,bc-c^2}{2\,bc} = \frac{a^2-(b^2-2\,bc+c^2)}{2\,bc}$$

$$= \frac{a^2-(b-c)^2}{2\,bc}$$

$$= \frac{(a+\overline{b-c})\ (a-\overline{b-c})}{2\,bc}.$$

$$\therefore\ bc\,.\,\text{Hav. A} = \frac{(a+\overline{b-c})\ (a-\overline{b-c})}{4},$$

or, $bc\,.\,\text{Hav. A} = \frac{1}{2}(a + \overline{b - c})\,.\,\frac{1}{2}(a - \overline{b - c})$

$\therefore$ Log. $b$ + Log. $c$ + Log. Hav. A $- 10 =$ Log. $\frac{1}{2}(a + \overline{b - c})$ + Log. $\frac{1}{2}(a - \overline{b - c})$

$\therefore$ Log. Hav. A $= 10 -$ (Log. $b$ + Log. $c$) + Log. $\frac{1}{2}(a+b-c)$ + Log. $\frac{1}{2}(a - \overline{b - c})$, (D)

Which, expressed in words, is Rule I.

## Proposition IV.

*To prove that* $a+b : a-b :: \text{Tan.} \frac{1}{2}(A+B) : \text{Tan.} \frac{1}{2}(A-B)$.

By Prop. (1), $\frac{a}{b} = \frac{\text{Sin. A}}{\text{Sin. B}}$.

Adding 1 to each side of the equation, we have—

$$\frac{a}{b} + 1 = \frac{\text{Sin. A}}{\text{Sin. B}} + 1,$$

$$\text{or,}\quad \frac{a+b}{b} = \frac{\text{Sin. A} + \text{Sin. B}}{\text{Sin. B}}. \qquad (1)$$

Subtracting 1 from each side, we have—

$$\frac{a}{b} - 1 = \frac{\text{Sin. A}}{\text{Sin. B}} - 1$$

$$\text{or,}\quad \frac{a-b}{b} = \frac{\text{Sin. A} - \text{Sin. B}}{\text{Sin. B}}. \qquad (2)$$

Dividing (1) by (2),

$$\frac{a+b}{a-b} = \frac{\text{Sin. A} + \text{Sin. B}}{\text{Sin. A} - \text{Sin. B}}$$

$$= \frac{2\,\text{Sin.} \frac{1}{2}(A+B)\ \text{Cos.} \frac{1}{2}(A-B)}{2\,\text{Cos.} \frac{1}{2}(A+B)\ \text{Sin.} \frac{1}{2}(A-B)} \qquad (24)\ \&\ (25)$$

$$= \text{Tan.} \frac{1}{2}(A+B) \,.\, \text{Cot.} \frac{1}{2}(A-B)$$

$$= \frac{\text{Tan.} \frac{1}{2}(A+B)}{\text{Tan.} \frac{1}{2}(A-B)}.$$

Or, $a+b : a-b :: \text{Tan.} \frac{1}{2}(A+B) : \text{Tan.} \frac{1}{2}(A-B)$, (E)

Which proves Rule III.

## Proposition V.

*Two sides and the included angle being given, to investigate a rule for finding the third side.*

By Prop. (II), $c^2 = a^2 + b^2 - 2ab \,.\, \text{Cos.C}$,

And since

$$\text{Cos.}^2\frac{C}{2}+\text{Sin.}^2\frac{C}{2}=1 \ (9); \text{ and } \text{Cos.}^2\frac{C}{2}-\text{Sin.}^2\frac{C}{2}=\text{Cos. C.} \quad (29)$$

$$\therefore \quad c^2=(a^2+b^2)\left(\text{Cos.}^2\frac{C}{2}+\text{Sin.}^2\frac{C}{2}\right)-2ab\left(\text{Cos.}^2\frac{C}{2}-\text{Sin.}^2\frac{C}{2}\right)$$

$$=(a-b)^2 \text{ Cos.}^2\frac{C}{2} + (a+b)^2 \text{ Sin.}^2\frac{C}{2}$$

(by multiplying out and collecting the terms)

$$=(a-b)^2 \text{ Cos.}^2\frac{C}{2}\left\{1+\left(\frac{a+b}{a-b}\right)^2 . \text{ Tan.}^2\frac{C}{2}\right\} \qquad (1)$$

Now, as the tangent of an angle may have any value, there will be an angle whose tangent is equal to $\frac{a+b}{a-b} . \text{Tan.}\frac{C}{2}$: let $\theta$ be such an angle.

$$\text{Then,} \qquad \text{Tan } \theta = \left(\frac{a+b}{a-b}\right) . \text{ Tan. } \frac{C}{2}. \qquad (2)$$

$$\therefore \text{ From (1), } c^2 = (a-b)^2 . \text{ Cos. } ^2\frac{C}{2}\{1+\text{Tan.}^2\theta\}$$

$$= (a-b)^2 . \text{ Cos.}^2\frac{C}{2} . \text{ Sec.}^2\theta$$

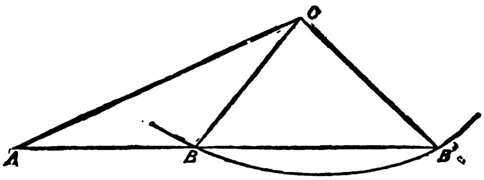
$$\text{Or,} \qquad c = (a-b) . \text{ Cos. } \frac{C}{2} . \text{ Sec. } \theta. \qquad (3)$$

Now, Tan. $\theta$ being determined from Eqn. (2), furnishes us with Sec. $\theta$, by which we are enabled to determine $c$ in Eqn. (3). This operation, expressed logarithmically, is Rule IV.

## On the Ambiguous Case.

The Ambiguous Case is when two sides and the angle opposite the *less* side are given.

In this instance there will be two solutions; for, let CAB be the given angle, AC one of the sides, with centre C, and distance $a$ (the value of the other side), describe the arc of a circle, cutting AB produced in B and B′ on the same side of A. Now each of the triangles CAB, CAB′ has all the data of the question, and therefore the solution is ambiguous.




## Right-Angled Triangles.

I. *When two sides are given to find the third side.*

Let ABC be a right-angled triangle, then, by Euclid, Book I, Prop. 47,

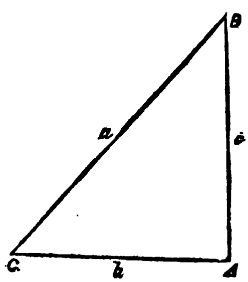
$$a^2 = b^2 + c^2 \qquad (1)$$

transposing—

$$b^2 = a^2 - c^2 = (a + c)\,(a - c) \qquad (2)$$

and

$$c^2 = a^2 - b^2 = (a + b)\,(a - b). \qquad (3)$$




Hence, when any two sides are given, the third may be found by one of the above formulæ.

II. *When c and b are given, to find a,* B, *and* C.

We have $\text{Tan. C} = \frac{c}{b}$,

or, $\text{Log. Tan. C} - 10^{*} = \text{Log. } c - \text{Log. } b$,

$\therefore$ $\text{Log. Tan. C} = 10 + \text{Log. } c - \text{Log. } b$,

which determines C, and $90° - \text{C} = \text{B}$.

And $\frac{a}{b} = \text{Sec. C}$, or, $a = b \,.\, \text{Sec. C}$,

$\therefore$ $\text{Log. } a = \text{Log. } b + \text{Log. Sec. C} - 10$.

III. *When a side and an angle are given, to find the remaining parts.*

*Ex.* Given $b$ and angle C, to find the other parts.

(1) To find $c$—

Put down the side *required*............ $c$
Under it put the *given* side............ $b$

Then write down the ratio of the given angle formed by the above expression } .. Tan. C.

Thus we have $\frac{c}{b} = \text{Tan. C}$,

or, $c = b \,.\, \text{Tan. C}$,

$\therefore$ $\text{Log. } c = \text{Log. } b + \text{Log. Tan. C} - 10$.

(2) To find $a$, we shall have by the above rule—

$\frac{a}{b} = \text{Sec. C}$, or, $a = b \,.\, \text{Sec. C}$,

$\therefore$ $\text{Log. } a = \text{Log. } b + \text{Log. Sec. C} - 10$,

and $\text{B} = 90° - \text{C}$.

Hence the triangle is completely solved.

---

* For the reason for subtracting 10 in this, and similar cases, see Chapter on Logarithms (note to the Table of Sines, &c.).

## Proposition VI.

*To express the sine of an angle in terms of the sides.*

First, by (9)—

$$\text{Sin.}^2\text{A} = 1 - \text{Cos.}^2\text{A} = (1 + \text{Cos. A})\,(1 - \text{Cos. A}).$$

But $\text{Cos. A} = \dfrac{b^2 + c^2 - a^2}{2bc}$ .. .. .. by (C)

$$\therefore\ \text{Sin.}^2\text{A} = \left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$= \frac{2bc + b^2 + c^2 - a^2}{2bc} \cdot \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{(b + c)^2 - a^2}{2bc} \cdot \frac{a^2 - (b - c)^2}{2bc}$$

$$= \frac{(a+b+c)\,(b+c-a)}{2bc} \cdot \frac{(a+b-c)\,(a-b+c)}{2bc}$$

$$= \frac{(a+b+c)\,(b+c-a)\,(a+b-c)\,(a-b+c)}{4b^2c^2}.(1)$$

Now, let $a + b + c = 2s$

then, $b + c - a = 2(s-a)$

$a - b + c = 2(s-b)$

$a + b - c = 2(s-c)$

$\therefore$ Substituting in (1), we have—

$$\text{Sin.}^2\text{A} = \frac{2s \,.\, \overline{2s - a} \,.\, \overline{2s - b} \,.\, \overline{2s - c}}{4b^2c^2}$$

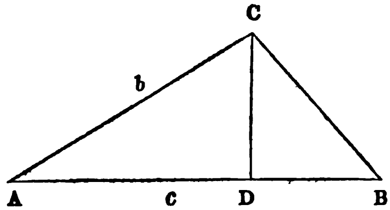
$$= \frac{4 \,.\, s \,.\, \overline{s - a} \,.\, \overline{s - b} \,.\, \overline{s - c}}{b^2c^2}$$

$$\therefore\ \text{Sin. A} = \frac{2}{bc} \cdot \sqrt{s \,.\, \overline{s - a} \,.\, \overline{s - b} \,.\, \overline{s - c}}.\ \text{(F)}$$

## PROPOSITION VII.

*Two sides and the included angle being given, to find the area of the triangle.*

*Ex.* *Given b, c, and A, to find the area.*




$$\text{Area} = \frac{\text{AB} \times \text{CD}}{2} = \frac{c \times \text{CD}}{2}.$$

But $\text{CD} = b \,.\, \text{Sin. A}$

$$\therefore \text{Area} = \frac{bc \,.\, \text{Sin. A}}{2}.$$

## PROPOSITION VIII.

*The three sides being given, to find the area.*

By Prop. VII. $\text{Area} = \frac{bc}{2} \cdot \text{Sin. A}.$

And by Prop. VI. $\text{Sin. A} = \frac{2}{bc} \sqrt{s \,.\, \overline{s-a} \,.\, \overline{s-b} \,.\, \overline{s-c}}$

$$\therefore \text{Area} = \frac{bc}{2} \times \frac{2}{bc} \cdot \sqrt{s \,.\, \overline{s-a} \,.\, \overline{s-b} \,.\, \overline{s-c}}$$

$$= \sqrt{s \,.\, \overline{s-a} \,.\, \overline{s-b} \,.\, \overline{s-c}}.$$

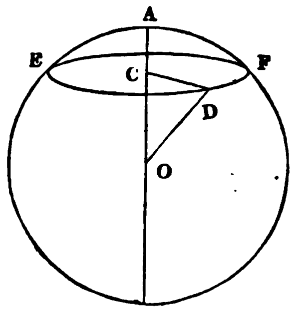
## CHAPTER IV.

### Spherical Trigonometry.

1. A *sphere* is a solid generated by the revolution of a semicircle about its diameter; and is such that all straight lines drawn from a point within it, called its centre, to the surface are equal.

2. The distance from the centre to the surface is called the *radius*: and any line passing through the centre, and terminated both ways by the surface, is called a diameter.

3. *Every section of a sphere made a plane is a circle.*




Let EDF be any such section of a sphere of which the centre is O. From O draw OC perpendicular to the cutting plane, and join CD and OD, D being any point in EDF. Then since OC is perpendicular to the plane, it is perpendicular to every line in the plane, and therefore to CD.

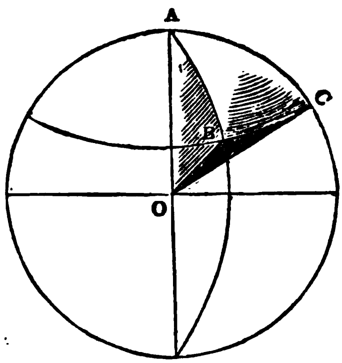
$$\therefore \quad OD^2 = OC^2 + CD^2,$$

$$\text{Or,} \quad CD^2 = OD^2 - OC^2.$$

But OD and OC are both constant quantities, therefore CD is also constant; or the section is a circle having C fc its centre.

4. A section of a sphere made by a plane passing through its centre is called a *great circle;* all other sections are called *small circles.*

5. A pole of a circle is equally distant from every point in the circumference of the circle, and a pole of a great circle is 90° distant from every point in the circumference of the great circle.




6. A spherical triangle is the portion of the surface of a sphere included between three arcs of great circles, as the triangle ABC. The angles being the inclinations of the planes of the great circles to each other, and the sides being measured by the angles which they subtend at the centre of the sphere, although only the triangle itself is usually shown in diagrams.

7. *The three angles of a spherical triangle are greater than two right angles, and less than six right angles, and the greater angle is opposite to the greater side.*

Of the three arcs forming the sides of a spherical triangle,

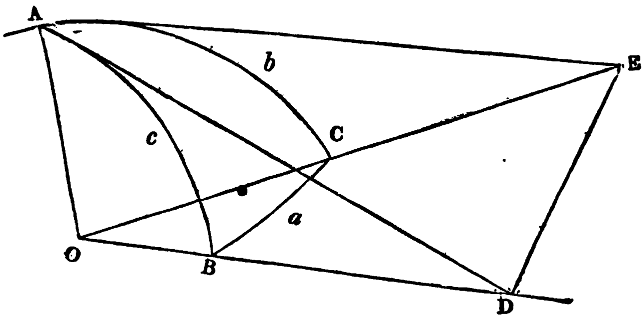
8. *Any one is less than a semicircle;*

9. *Any one is less than the sum of the other two;*

10. *And the sum of the three is less than the circumference of a great circle.*

11. *Proof of the fundamental formula in spherical trigonometry, viz., that*

$$\text{Cos. A} = \frac{\text{Cos. } a - \text{Cos. } b \text{ . Cos. } c}{\text{Sin. } b \text{ . Sin. } c}.$$




Let ABC be a spherical triangle, O the centre of the sphere; draw tangents to meet OC, OB produced in D and E, and join AO and DE.

Then OAD, OAE are right angles (Eucl. III, Prop. 18), and DAE being a plane triangle, we have by Prop. II, Ch. III,

$$\text{DE}^2 = \text{DA}^2 + \text{AE}^2 - 2\,\text{DA . AE Cos. DAE}.$$

And in the plane triangle DOE, we have, in like manner,

$$\text{DE}^2 = \text{DO}^2 + \text{OE}^2 - 2\,\text{DO . OE . Cos. DOE}.$$

Equating these two equal values of $\text{DE}^2$, we have,

$$\text{DA}^2 + \text{AE}^2 - 2\,\text{DA . AE Cos. DAE}$$
$$= \text{DO}^2 + \text{OE}^2 - 2\,\text{DO . OE Cos. DOE}.$$

Transposing and placing $DA^2$ after $DO^2$, and $AE^2$ after $OE^2$,

$$- 2\,DA\,.\,AE\ \text{Cos.}\ DAE$$
$$= DO^2 - DA^2 + OE^2 - AE^2 - 2\,DO\,.\,OE\ \text{Cos.}\ DOE.$$

But $DO^2 - DA^2 = AO^2$, and $OE^2 - AE^2 = AO^2$, since OAD and OAE are right angles.

$$\therefore\quad - 2\,DA\,.\,AE\,.\,\text{Cos.}\ DAE$$
$$= AO^2 + AO^2 - 2\,DO\,.\,OE\ \text{Cos.}\ DOE$$
$$= 2\,AO^2 - 2\,DO\,.\,OE\ \text{Cos.}\ DOE$$
$$= 2\,AO\,.\,AO - 2\,DO\,.\,OE\,.\,\text{Cos.}\ DOE.$$

Dividing by $2\,DO\,.\,OE$,

$$-\frac{DA\,.\,AE}{DO\,.\,OE}\,.\,\text{Cos.}\ DAE = \frac{AO\,.\,AO}{DO\,.\,OE} - \text{Cos.}\ DOE.$$

But $\dfrac{AE}{OE} = \text{Sin.}\ AOE = \text{Sin.}\ b$, $\dfrac{AD}{DO} = \text{Sin.}\ AOD = \text{Sin.}\ c$,

$$\frac{AO}{DO} = \text{Cos.}\ c, \qquad \frac{AO}{OE} = \text{Cos.}\ b,$$

$$\text{Cos.}\ DAE = \text{Cos.}\ A, \quad \text{and} \quad \text{Cos.}\ DOE = \text{Cos.}\ a.$$

$\therefore$ Making these substitutions,

$$-\text{Sin.}\ b\,.\,\text{Sin.}\ c\,.\,\text{Cos.}\ A = \text{Cos.}\ b\,.\,\text{Cos.}\ c - \text{Cos.}\ a,$$

Whence $\text{Cos.}\ A = \dfrac{\text{Cos.}\ a - \text{Cos.}\ b\,.\,\text{Cos.}\ c}{\text{Sin.}\ b\,.\,\text{Sin.}\ c}.$ (A)

12. *To investigate a formula for finding an angle when the three sides are given.*

By formula (A).

$$\text{Cos. A} = \frac{\text{Cos. } a - \text{Cos. } b \,.\, \text{Cos. } c}{\text{Sin. } b \,.\, \text{Sin. } c}.$$

$$\therefore \quad 1 - \text{Cos. A} = 1 - \frac{\text{Cos. } a - \text{Cos. } b \,.\, \text{Cos. } c}{\text{Sin. } b \,.\, \text{Sin. } c},$$

$$\text{Or, Vers. A} = \frac{\text{Sin. } b \,.\, \text{Sin. } c - \text{Cos. } a + \text{Cos. } b \,.\, \text{Cos. } c}{\text{Sin. } b \,.\, \text{Sin. } c}$$

$$= \frac{\text{Cos. } b \,.\, \text{Cos. } c + \text{Sin. } b \,.\, \text{Sin. } c - \text{Cos. } a}{\text{Sin. } b \,.\, \text{Sin. } c}$$

$$= \frac{\text{Cos. } (b-c) - \text{Cos. } a}{\text{Sin. } b \,.\, \text{Sin. } c} = -\frac{\text{Cos. } (b-c) + \text{Cos. } a}{\text{Sin. } b \,.\, \text{Sin. } c}$$

$$\therefore \; 2 \text{ Hav. A} = \frac{2 \text{ Sin. } \frac{(b-c)+a}{2} \,.\, \text{Sin. } \frac{(b-c)-a}{2}}{\text{Sin. } b \,.\, \text{Sin. } c}, \text{ by } (23)$$

$$\therefore \quad \text{Hav. A} = \frac{\sqrt{\text{Hav. } (\overline{b-c}+a)} \quad \sqrt{\text{Hav. } (\overline{b-c}-a)}}{\text{Sin. } b \,.\, \text{Sin. } c}. \text{ by } (32)$$

Or, in Logs.,

$$\text{Log. Hav. A} - 10 = \text{Log. Cosec. } b - 10 + \text{Log. Cosec. } c$$
$$-10 + \tfrac{1}{2}\{\text{Log. Hav. } (\overline{b-c}+a) - 10\}$$
$$+ \tfrac{1}{2}\{\text{Log. Hav. } (\overline{b-c}-a) - 10\}$$
$$\therefore \; \text{Log. Hav. A} = \text{Log. Cosec. } b + \text{Log. Cosec. } c$$
$$+ \tfrac{1}{2}\text{Log. Hav. } (\overline{b-c}+a) + \tfrac{1}{2} \text{ Log. Hav. } (\overline{b-c}-a) - 20,$$

Which, expressed in words, is Rule I, Spherical Trigonometry.

13. *To investigate the rule for finding the third side of a spherical triangle, when two sides and the included angle are given.*

By formula (A).

$$\text{Cos. A} = \frac{\text{Cos. } a - \text{Cos. } b \, . \, \text{Cos. } c}{\text{Sin. } b \, . \, \text{Sin. } c}$$

$$\therefore \; 1 - \text{Cos. A} = 1 - \frac{\text{Cos. } a - \text{Cos. } b \, . \, \text{Cos. } c}{\text{Sin. } b \, . \, \text{Sin. } c}.$$

$$\text{Or,} \quad \text{Vers. A} = \frac{\text{Sin. } b \, . \, \text{Sin. } c - \text{Cos. } a + \text{Cos. } b \, . \, \text{Cos. } c}{\text{Sin. } b \, . \, \text{Sin. } c}$$

$$= \frac{(\text{Cos. } b \, . \, \text{Cos. } c + \text{Sin. } b \, . \, \text{Sin. } c) - \text{Cos. } a}{\text{Sin. } b \, . \, \text{Sin. } c}$$

$$= \frac{\text{Cos. } (b - c) - \text{Cos. } a}{\text{Sin. } b \, . \, \text{Sin. } c}.$$

$$\therefore \; \text{Sin. } b \, . \, \text{Sin. } c \, . \, \text{Vers. A} = \text{Cos. } (b - c) - \text{Cos. } a.$$

Transposing, $\text{Cos. } a = \text{Cos. } (b - c) - \text{Sin. } b \; \text{Sin. } c \; \text{Vers. A}.$

Subtracting each side from (1),

$$1 - \text{Cos. } a = 1 - \text{Cos. } (b - c) + \text{Sin. } b \, . \, \text{Sin. } c \; \text{Vers. A}.$$

Or, $\quad \text{Vers. } a = \text{Vers. } (b - c) + \text{Vers. } \theta.$

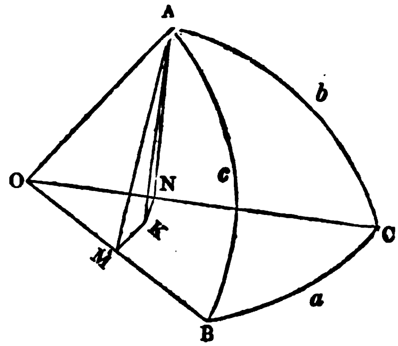
Where $\theta$ is determined from the formula

$$\text{Hav. } \theta = \text{Sin. } b \, . \, \text{Sin. } c \; \text{Hav. A}.$$

Which, in words, is Rule II.

14. *To prove the rule of sines in spherical trigonometry, viz.: that* Sin. B : Sin. C :: Sin. $b$ : Sin. $c$.

Let ABC be a spherical triangle; drop the perpendicular AK on the plane BOC, and from K, draw KM, KN perpendicular to the edges OB, OC, and join AM, AN. Then it is evident that angle AMK = B and ANK = C, and the triangles AMK, ANK being right-angled, we have—




$$AK = AM \,.\, \text{Sin. AMK} = AM \,.\, \text{Sin.B}; \;\&\; AK = AN \,.\, \text{Sin.C.}$$

$$\therefore \; AM \,.\, \text{Sin. B} = AN \,.\, \text{Sin. C.} \qquad (1)$$

But also AMO, ANO are right-angled triangles, AMO and ANO being the right angles.

$$\therefore \; AM = AO \text{ Sin. AOM} = AO \,.\, \text{Sin. } c,$$

$$AN = AO \text{ Sin. AON} = AO \,.\, \text{Sin. } b.$$

∴ Substituting for AM, and AN in Eq. (1), we have—

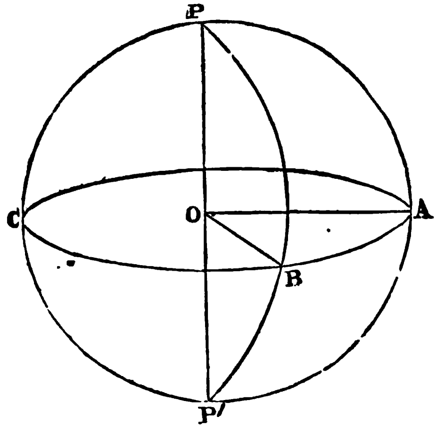
$$AO \,.\, \text{Sin. } c \,.\, \text{Sin. B} = AO \,.\, \text{Sin. } b \,.\, \text{Sin. C.}$$

$$\therefore \; \text{Sin. } c \,.\, \text{Sin. B} = \text{Sin. } b \,.\, \text{Sin. C},$$

$$\text{or,} \quad \text{Sin. B : Sin. C :: Sin. } b : \text{Sin. } c,$$

Which is the rule.

15. *To prove that the angle* APB, *at either pole of the great circle* ABC, *is measured by the arc* AB *included between the two great circles* PAP′, PBP′.



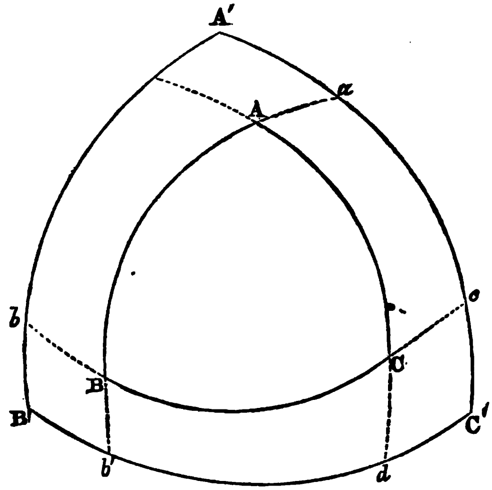

Let ABC be a great circle, P one of its poles, APB the angle included between the two great circles PAP′, PBP′; then this angle will be measured by the arc AB.

Take O the centre of the sphere, and join OA, OB. Then, because the arc PA lies in the plane POA, and the arc PB in the plane POB, the angle AOB between these two planes is equal to the angle APB (by the definition of a spherical angle). But AB is the same fraction of the circumference of a circle as the angle AOB is of two right angles; therefore, as AB measures the angle AOB, it also measures APB, which is equal to AOB.

16. *On the Supplemental Triangle.*

Let ABC be a spherical triangle, and with A, B, C as poles, describe the arcs of great circles B′C′, C′A′, A′B′; then these sides of the triangle A′B′C′ will be the supple-

ments of the angles A, B, C, and the angles, A′, B′, C′, will be the supplements of BC, AC, AB.




For $A'C' = A'a + C'c + ac$

$= 90^\circ - ac + 90^\circ - ac + ac$

$= 180^\circ - ac = 180^\circ - B.$

Since $ac$ measures B (see preceding Art.)

$\therefore$ A′C′ is the supplement of B.

In like manner we may prove B′C′, and A′B′ to be the supplements of A and C.

So also will A′ be the supplement of BC.

For $bc = Bb + Cc + BC$

$= 90^\circ - BC + 90^\circ - BC + BC$

Or, $A' = 180^\circ - BC$

*i.e.*, A′ is the supplement of BC.

And, for similar reasons, C′ and B′ will be the supplements of AB and AC. So that the sides and angles of the two triangles are supplements, the one of the other.

## CHAPTER V.

### On Logarithms.

*Def.*—The logarithm of a number to a given base is the index of the power to which the base must be raised to be equal to the number; thus, in the equation $a^x = \text{N}$, $x$ is called the logarithm of N to the base $a$. The base most generally used is 10, so that we may say, the logarithm of a number is that power to which 10 must be raised to produce the number.

Thus, since
$$\begin{aligned} 10^0 &= 1, \\ 10^1 &= 10, \\ 10^2 &= 100, \\ 10^3 &= 1000, \\ \&c. &= \&c., \end{aligned}$$

0, 1, 2, 3, &c., are the Logs. of 1, 10, 100, 1000, &c. Hence we see that the Logs. of all numbers between 0 and 10 will be greater than 0, and less than 1, and of all numbers between 10 and 100, greater than 1, and less than 2. Logarithms will, therefore, be represented by the numbers 0, 1, 2, 3, &c., with some decimals after them; the decimal part being called the *mantissa*,* and the integral part, the *characteristic* or *index*; and it will be seen from above that the characteristic or index will be *less by one* than the number of figures in the given number.

Again,
$$\begin{aligned} \cdot 1 &= \frac{1}{10} = \frac{1}{10^1} = 10^{-1}, \\ \cdot 01 &= \frac{1}{100} = \frac{1}{10^2} = 10^{-2}, \\ \cdot 001 &= \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}, \\ \&c. &= \&c. \end{aligned}$$

Therefore, in decimal numbers, the characteristic is *negative*, and *greater by one* than the number of the ciphers after the decimal point.

---

* *Mantissa*, a Latin word, meaning a handful thrown in over and above the exact weight.

### *Proofs of the principal properties of Logarithms.*

1. The Log. of the *product* of two numbers is the *sum* of the Logs. of the numbers.

Let the numbers be N and M, and

let $$x = \text{Log. N}, \quad y = \text{Log. M},$$

or, $$\text{N} = 10^x, \quad \text{M} = 10^y$$

$$\therefore \ \text{M} \times \text{N} = 10^x \times 10^y = 10^{x+y} \,*$$

$\therefore$ $x + y$ is the Log of M × N, by the definition

*i.e.,* $$\text{Log. (M} \times \text{N)} = x + y = \text{Log. N} + \text{Log. M}.$$

2. The Log. of the *quotient* of two numbers is the *difference* of the Logs. of the numbers.

For, $$\frac{\text{N}}{\text{M}} = \frac{10^x}{10^y} = 10^{x-y},$$

$$\therefore \ x - y \text{ is the Log. of } \frac{\text{N}}{\text{M}}$$

*i.e.,* $$\text{Log.} \left(\frac{\text{N}}{\text{M}}\right) = x - y = \text{Log. N} - \text{Log. M}.$$

3. The Log. of *any power* of a number is equal to the Log. of the number multiplied by the *index of the power*.

For, since $$\text{N} = 10^x,$$

$$(\text{N})^r = (10^x)^r = 10^{rx}$$

$$\therefore \ rx \text{ is the Log. of } (\text{N})^r$$

*i.e.,* $$\text{Log. } (\text{N})^r = rx = r \text{ Log. N}.$$

4. The Log. of the *root* of a number is the Logarithm of the number divided by the quantity expressing the *root*.

For, since $$\text{N} = 10^x,$$

$$(\text{N})^{\frac{1}{r}} = (10^x)^{\frac{1}{r}} = 10^{\frac{x}{r}}$$

$$\therefore \ \frac{x}{r} \text{ is the Log of } (\text{N})^{\frac{1}{r}}$$

$$\therefore \ \text{Log. N}^{\frac{1}{r}} = \frac{x}{r} = \frac{\text{Log. N.}}{r}$$

Hence it appears, that by means of a table of Logs., *multiplication* may be performed by *addition ; division* by *subtraction ; involution,* by *multiplication ;* and *evolution,* by *division.*

---

* *Todhunter's Algebra,* Art. 317.

# CHAPTER VI.

## On the Use of the Tables of Logs.

### RULE I.

*To find the index of the Log. of a whole or mixed number.*

The index is *one less* than the number of the figures in the *integral* part of the number. (See page 45).

| | | | |
|---|---|---|---|
| Thus, the index of | 1 | is | 0, |
| | 12 | „ | 1, |
| | 123 | „ | 2, |
| | 1234 | „ | 3, |
| | 1234567 | „ | 6, &c. |

Conversely: The digits in a number will be *one more* than the *index* of its Log.

### RULE II.

*To find the index of the Log. of a decimal fraction.*

The index is *negative*, and one *greater* than the number of the ciphers following the decimal point. (See page 45).

| | | | |
|---|---|---|---|
| Thus, the index of | ·1 | is | $\bar{1}$ |
| | ·01 | „ | $\bar{2}$ |
| | ·0001 | „ | $\bar{4}$, &c. |

Conversely: The number of ciphers, *after* the decimal point, will be one *less* than the characteristic or index of the Logarithm of the number.

## To Explain the Use of the Logarithmic Tables.

(*The Tables here referred to are those edited by Dr. Inman.*)

I. *To take out the Log. of a natural number.*

If the number consist of not more than four figures, the decimal part of its Log. may be taken out at once by inspection from table at page 217, and the index is to be prefixed according to Rule I. Thus, Log. 2 = 0·301030; Log. 250 = 2·397940; Log. 1280 = 3·107210; &c.

II. *When there are more than four figures in the number, we must proceed as in the following example:—*

*To find the Log. corresponding to the natural number,* 12834568.

| | | | | |
|---|---|---|---|---|
| From the accompanying Table, Log. 1283 | | | = | ·108227 |
| Parts for 4, in right-hand column | | | .. | 135 |
| „ 5 | „ | „ | .. | 169* |
| „ 6 | „ | „ | .. | 203 |
| „ 8 | „ | „ | .. | 270 |
| | | | | ·108381200 |

The number of figures in the given number being 8, the characteristic or index will be 7 (Rule I.); thus Log. 12834568 = 7·108381, omitting the three last figures.

---

* The figures are to be set *one* to the right each time.

*Extract from the Table* (p. 219, Inman's Tables).

| No. | Log. | Part. |
|---|---|---|
| 1280 | ·107210 | 000 |
| 1 | ·107549 | 034 |
| 2 | ·107888 | 067 |
| 3 | ·108227 | 101 |
| 4 | ·108565 | 135 |
| 5 | ·108903 | 169 |
| 6 | ·109241 | 203 |
| 7 | ·109578 | 237 |
| 8 | ·109916 | 270 |
| 9 | ·110253 | 304 |

III. *To find the number corresponding to* Log. 6·108381.

Look for the *decimal* part of the Log. in the Tables, and if found exactly, the number opposite will be that required, adding, if necessary, as many ciphers as will make it to consist of a number of figures exceeding the index by *one*.

| | | | | |
|---|---|---|---|---|
| But if you cannot exactly find | | 108381 | | |
| Put down the next less | .. | 108227 | = | 1283 |
| And take the difference | .. | 154 | | |
| Parts next less than 154 | .. | 135 | = | 4 |
| | | 190* | | |
| ,, ,, 190 | .. | 169 | = | 5 |
| | | 210 | | |
| ,, ,, 210 | .. | 203 | = | 6 |
| | | Sum | | 1283456 |

* A cipher is added after each subtraction to compensate for the figures omitted in the Log. after the sixth decimal place.

Thus, Log. 6·108381 = 1283456.

The index being 6, the figures in the number must be 7. (by Rule I.)

Similarly,

Log. ·1283 = $\bar{1}$·108227; Log. ·001289 = $\bar{3}$·110253.

IV. *To take out the Log. Sine, Cosine, Tangent, &c., of any angle.*

In table at page 37, look for the degree at the *top* of the page, and the minutes and seconds at the left-hand side, if the angle be *less* than 45°; but for the degrees at the *bottom*, and the minutes and seconds at the *right-hand* side, if the angle be *greater* than 45°.

Thus, Log. Sin. 30° 10′ 15″ = 9·701205; and Log. Tan. 50° 6′ 30″ = 10·077855.

But if the angle be required to the *nearest second*, proceed as in the following example:—

To find Log. Sin. 30° 20′ 41″, and Log. Cos. 40° 30′ 52″ to the nearest second.

Under the given angle, put down two between which it lies, taken at the nearest half-minutes; take the difference between the first and second, and second and third angles; multiply the difference of the Logs. of the last two by the *first* difference, and divide the product by the *second* difference; the result will be the parts to be added to, or subtracted from, the first Log., according as the *Logs.* are increasing or decreasing; thus:—

To find Log. Sin. 30° 20′ 41″.

| | | | |
|---|---|---|---|
| | 30° 20′ 41″ | | |
| | 30 20 30 | Log. Sin. | 9·703425 |
| | 30 21 0 | Log. Sin. | 9·703533 |
| 1st difference | 11 | | 108 |
| 2nd difference | 30 | | 11 |
| | | 30 | 1188 |
| | | Parts for 11″ = | 39·6 |

| | | | |
|---|---|---|---|
| Log. Sin. $30^\circ\ 20'\ 30''$ | | = | 9·703425 |
| Parts for | 11 | = | 39 |
| Log. Sin. 30 20 41 | | = | 9·703464 |

The parts are *added*, because the Logs. are *increasing*.

To find Log. Cos. $40^\circ\ 30'\ 52''$.

| | | | |
|---|---|---|---|
| | $40^\circ\ 30'\ 52''$ | | |
| | 40 30 30 | Log. Cos. | 9·880992 |
| | 40 31 0 | Log. Cos. | 9·880938 |
| 1st difference | 22 | | 54 |
| 2nd difference | 30 | | 22 |
| | | | 108 |
| | | | 108 |
| | | 30 | 1188 |
| | | Parts for $22''$ = | 39·6 |

| | | | |
|---|---|---|---|
| Log. Cos. $40^\circ\ 30'\ 30''$ | | = | 9·880992 |
| Parts for | 22 | = | 39 |
| Log. Cos. 40 30 52 | | = | 9·880953 |

The parts are *subtracted*, because the Logs. are *decreasing*.

The Log. Tangents, Log. Secants, &c., of Angles are found in a similar manner.

If the Sine, Tangent, or Secant of an angle greater than $90^\circ$ be required, diminish the angle by $90^\circ$, and take the Cosine, Cotangent, or Cosecant of the remainder. Thus, Sin. $120^\circ$ = Cos. $30^\circ$, Tan. $120^\circ$ = Cot. $30^\circ$, Sec. $120^\circ$ = Cosec. $30^\circ$, &c.

V. *To take out an angle corresponding to a given Log.*

Under the given Log. put down the two between which it lies (taken out at the nearest half-minutes). Take the difference between the first and second, and the second and third; multiply the first difference by 30, and divide the product by the second difference; the result will be the

number of seconds to be *added* to the angle corresponding to the first Log.

**Ex.** To find the angle corresponding to Log. Sin. 9·703465.

| | | | |
|---|---|---|---|
| | 9·703465 | | |
| | 9·703425 | .. | 30° 20′ 30″ |
| | 9·703533 | .. | 30 21 0 |
| | | | 30 |
| 1st difference | 40 | | 40 |
| 2nd difference | 108 | | 108 \| 1200 \| 11 Corr. |
| | | | 1188 |

| | | |
|---|---|---|
| Angle | .. | 30° 20′ 30″ |
| Corr. | .. | 11 |
| Answer | | 30 20 41 |

**Ex.** To find the angle corresponding to Log. Cosine 9·880953.

| | | | |
|---|---|---|---|
| | 9·880953 | | |
| | 9·880992 | .. | 40° 30′ 30″ |
| | 9·880938 | .. | 40 31 0 |
| | | | 30 |
| 1st difference | 39 | | 39 |
| 2nd difference | 54 | | 54 \| 1170 \| 22 Corr. |
| | | | 108 |
| | | | 90 |

| | | |
|---|---|---|
| Angle | .. | 40° 30′ 30″ |
| Corr. | .. | 22 |
| Answer | | 40 30 52 |

The angles corresponding to Log. Tangents, Log. Secants, &c., may be found in a similar way.

VI. *To take out the Versine of angle to the nearest second* (Inman's Tables, page 248).

**Ex.** Find Vers. 52° 20′ 30″.

| | |
|---|---|
| Look for the degrees at the top of the page, and the minutes at the left-hand side, and at the angle of meeting you will find .. .. .. | 0388933 |
| Again, look for the degrees at the top of the next page, and the seconds at the left side, and under the column belonging to 52°, marked 0′,* and in a line with 40″, you will find .. .. | 153 |
| Which, added to the first part, gives the Vers. required .. .. .. .. .. .. | 0389086 |

VII. *To find the angle corresponding to a given Versine.*

**Ex.** *Find the angle corresponding to* 0389086.

| | | |
|---|---|---|
| Put down the given Versine .. | 0389086 | |
| Under it put the next less in the table, with the corresponding angle | 0388933 .. | 52° 20′ 0″ |
| Difference .. .. | 153 | |
| Put your pen on 52°, in the right-hand page, and move it down the column marked 0′, till you come to 153. The seconds opposite 153 in the left-hand column are .. .. | | 40 |
| Therefore, the angle required is | | 52 20 40 |

* This column is to be entered when the minutes in the angle are less than 30, and the next column when they are greater than 30.

## The Table of Haversines.

This table is found at page 217 (Inman's), and is so simple in its arrangement as to require but little explanation. The Logs. are computed for every 15″, but should greater accuracy be required, the Log. Haversine of an angle to the nearest second may be found in the same manner as the Log. Sine, Log. Cosine, &c.

VIII. *To take out the number of hours, minutes, and seconds, corresponding to a given Haversine.*

**Ex.** *To find the Time corresponding to Log. Haversine* 9·056211.

At the top of the page in which the Log. occurs you will find 2 hours, and vertically over the column to which the Log. belongs is 37 minutes, while in the column of seconds at the left side of the page and in a line with the Log. will be found 44 sec.

Thus, the time required is $2^h\ 37^m\ 44^s$.

## Exercises on the Tables of Logarithms.

*On the Table of Natural Numbers* (p. 217).

| | | | |
|---|---|---|---|
| 1. Find the Logarithm of | 876. | *Ans.* | 2·942504. |
| ,, ,, | 3564. | *Ans.* | 3·551938. |
| ,, ,, | 1725674. | *Ans.* | 6·236958. |
| ,, ,, | 987654267. | *Ans.* | 8·994605. |
| ,, ,, | ·0876. | *Ans.* | $\bar{2}$·942504. |
| ,, ,, | ·000856736. | *Ans.* | $\bar{4}$·932847. |
| ,, ,, | ·003564. | *Ans.* | $\bar{3}$·551938. |
| ,, ,, | 753·4125. | *Ans.* | 2·877033. |
| ,, ,, | 109872·5. | *Ans.* | 5·040889. |
| ,, ,, | $\frac{3}{4}$, or ·75. | *Ans.* | $\bar{1}$·875061. |
| ,, ,, | $\frac{5}{8}$, or ·625 | *Ans.* | $\bar{1}$·795880. |
| ,, ,, | $210\frac{1}{2}$, or 210·5 | *Ans.* | 2·323252. |
| ,, ,, | $313\frac{1}{4}$, or 313·25 | *Ans.* | 2·495891. |
| ,, ,, | $33·3\dot{3}$ | *Ans.* | 1·522878. |

**2. Find the natural numbers corresponding to the following Logs:—**

| | |
|---|---|
| 1·567342. | *Ans.* 36·92676. |
| 3·892465. | *Ans.* 7806·671. |
| 0·927634. | *Ans.* 8·465139. |
| $\bar{1}$·186324. | *Ans.* 0·1535763. |
| $\bar{4}$·321546. | *Ans.* 0·0002096. |
| $\bar{2}$·532165. | *Ans.* 0·0340537. |
| 5·456789. | *Ans.* 286278·526. |
| 6·012345. | *Ans.* 1028830·707. |
| 7·103412. | *Ans.* 12688523·202. |
| $\bar{3}$·413715. | *Ans.* ·0025924. |
| $\bar{5}$·234567. | *Ans.* ·000017162. |
| $\bar{6}$·371806. | *Ans.* ·000002354. |

*On the Table of Sines, Cosines, &c.*

To find the Log. Sine, &c., of any angle.

| | | | |
|---|---|---|---|
| 1. Find the Sine of | 6° 36′ 27″. | *Ans.* | 9·060951. |
| 2. Find the Cosine of | 31 28 42. | *Ans.* | 9·930867. |
| 3. Find the Sine of | 86 3 17. | *Ans.* | 9·998969. |
| 4. Find the Cosine of | 57 32 36. | *Ans.* | 9·729701. |
| 5. Find the Tangent of | 40 20 10. | *Ans.* | 9·928983. |
| 6. Find the Secant of | 79 2 5. | *Ans.* | 10·720757. |

3. *Having given Log. Sine, Log. Cosine, &c., to find the angle to the nearest second.*

1. Given Log. Sine $= 9{\cdot}356241$; find the angle.
   *Ans.* $13^\circ\ 7'\ 37''$.
2. Given Log. Cosine $= 9{\cdot}873241$; find the angle.
   *Ans.* $41^\circ\ 40'\ 50''$.
3. Given Log. Tan. $= 9{\cdot}796342$; find the angle.
   *Ans.* $32^\circ\ 1'\ 58''$.
4. Given Log. Hav. $= 9{\cdot}818753$; find the angle.
   *Ans.* $108^\circ\ 31'\ 4''$.
5. Given Log. Tan. $= 9{\cdot}928983$; find the angle.
   *Ans.* $40^\circ\ 20'\ 10''$.
6. Given Log. Sec. $= 10{\cdot}720757$; find the angle.
   *Ans.* $79^\circ\ 2'\ 5''$.

*Obs.* All the Log. Sines, Cosines, &c., in the Tables are *increased* by 10, in order to avoid the use of *negative* characteristics. For example, Sin. $30^\circ$ equals $\frac{1}{2}$, or $\cdot 5$ (see Chap. I, Art. 19), therefore, Log. Sin. $30^\circ =$ Log. $\cdot 5$ or $\bar{1}{\cdot}698970$. As it would be inconvenient to place it in the Tables in this form, 10 is added to its characteristic, when it becomes $9{\cdot}698970$; and so in other cases. Care must be taken, therefore, when we are using the Log. Sines, Cosines, &c., from the Tables, to reject 10 for every one used. Thus, suppose it is required to find by Logs. the value of $x$ in the equation $x = a\,.\,\text{Sin. B}$. Then we have

$$\text{Log. } x = \text{Log. } a + \text{Log. Sin. B} - 10,$$

subtracting 10 from Log. Sin. B, for the reason above mentioned.

If the equation were $x = a\,.\,\text{Sin.}^2\,\text{B}$, then we should have

$$\text{Log. } x = \text{Log. } a + 2(\text{Log. Sin. B} - 10),$$

or, $\text{Log. } x = \text{Log. } a + 2\ \text{Log. Sin. B} - 20$, and so on.

## Multiplication by Logarithms.

*Rule.* Add together the Logs. of the *factors;* the result will be the Log. of their *product,* which take from the Tables in the usual way.

1. Find the product of 5632·1 and 32·56458.

Log. 5632·1 = 3·750671

Log. 32·56458 = 1·512744

5·263415 = 183406·75 *Ans.*

2. Find the product of 3846 and ·056214.

Log. 3846 = 3·585009

Log. ·056214 = $\bar{2}$·749845

2·334854 = 216·199 *Ans.*

The index, 2 = $\bar{2}$ + 1 (carried) + 3 = $\bar{1}$ + 3.

Find by Logs. the product

3. Of 24·13 × 6·052. *Ans.* 146·0347.

4. Of 49·51 × 283·605. *Ans.* 14041·25.

5. Of ·007461 × ·3351767. *Ans.* ·00250075.

6. Of 58 × 1·405 × 840. *Ans.* 68451·6.

Find the continued product

7. Of 240 × ·24 × ·0024 × 2400. *Ans.* 331·77.

8. Of 78524 × ·00079 × 24 × ·0000036. *Ans.* ·00535972.

## Division by Logarithms.

*Rule.* Subtract the Log. of the *divisor* from that of the *dividend;* the result will be the Log. of the *quotient.*

Find by Logs. the quotient

1. Of $35274 \div 5678$. *Ans.* 6·2124.

2. Of $11 \div {·}3929$. *Ans.* 27·997.

3. Of ${·}9649 \div 35{·}00583$. *Ans.* ·027564.

4. Of ${·}26439 \div {·}28629$. *Ans.* ·923202.

Find the value

5. Of $\frac{19}{72}$.* *Ans.* ·2639.

6. Of $\frac{19}{{·}72}$. *Ans.* 26·39.

7. Of $\frac{{·}0345}{928}$. *Ans.* ·00003717.

8. Of $\frac{74}{{·}0000435}$. *Ans.* 1701155·12.

---

* As $\frac{19}{72}$ expresses $19 \div 72$, we have only to subtract Log. 72 from Log. 19 to obtain the Log. value of the fraction.

## Involution by Logarithms.

***Rule.*** **To raise a number to any *power*, multiply the *Log.* of the number by the *index* of the power; the result will be the Log. of the required *power*; if the index be a fraction, multiply the Log. by the numerator, and divide the product by the denominator.**

### Examples.

1. **Find the 4th power of 2.**

$$\begin{array}{r} \text{Log. } 2 = \cdot301030 \\ 4 \\ \hline 1\cdot204120 \end{array}$$

*Ans.* 16.

2. **Find the value of $(\cdot 2)^{\frac{4}{5}}$.**

$$\begin{array}{r} \text{Log. } 2 = \bar{1}\cdot301030 \\ 4 \\ \hline 5\,)\,\bar{3}\cdot204120 \\ \hline \bar{1}\cdot440824 \end{array}$$

*Ans.* ·275.

In *Ex.* (2) the index $\bar{3} = \bar{1} \times 4 + 1$ (carried).
See also *Obs.* on next page.

3. Find the 5th power of 11. *Ans.* 161051.
4. Find the cube of 196·3. *Ans.* 7564151.
5. Find the cube of ·008 *Ans.* ·000000512.

$$\begin{array}{r} \text{Log. } \cdot008 = \bar{3}\cdot903090 \\ 3 \\ \hline \bar{7}\cdot709270 \end{array}$$

The index $\bar{7} = \bar{3} \times 3 + 2$ (carried).

6. Find the square of ·08567. *Ans.* ·007339.

Find by Logs. the value of each of the following expressions:

7. $(\cdot204)^{8}$. *Ans.* ·00000299.
8. $(\cdot975)^{200}$. *Ans.* ·006324.
9. $(\cdot096)^{\frac{5}{9}}$. *Ans.* ·272.
10. $(\cdot472)^{\frac{4}{9}}$. *Ans.* ·7162.
11. $(\cdot2)^{\frac{11}{12}}$. *Ans.* ·2287.
12. $(2\cdot02)^{3\cdot3}$. *Ans.* 10·17.

## Evolution by Logarithms.

***Rule.*** **To find the *root* of a given number, divide the *Log.* of the number by the *index* of the *root;* the result will be Log. of the required *root.***

### Example.

Find the cube root of ·08.

$$3 \,\underline{|\, \bar{2}\cdot903090\ldots\ (\text{Log. of } \cdot08)}$$
$$\bar{1}\cdot634363$$

*Ans.* ·4309.

***Obs.*** **In a case of this kind, in which the index is negative, and *not* divisible by the number expressing the root, it is necessary to increase the *negative* index by as many units as will render it divisible, and then carry so many to the decimal. Thus, in preceding example, $\bar{2} = \bar{3} + 1$, and, therefore,**

$$3\,\underline{|\,\bar{3} + 1\cdot903090}$$

**is** $\bar{1} + \cdot634363$ or $\bar{1}\cdot634363$.

### Examples.

Find by Logs.

1. The square root of 3·621409. *Ans.* 1·903.
2. The cube root of 3852. *Ans.* 15·675.
3. The 5th root of 24871·53. *Ans.* 7·57077.
4. The square root of ·00780908. *Ans.* ·0883688.
5. The 19th root of ·00123456. *Ans.* ·702944.

Find by Logs. the value, &c.,

6. Of $\sqrt[3]{3} \times \sqrt[4]{4} \times \sqrt[5]{5}$. *Ans.* 2·814.
7. Of $\sqrt[10]{\cdot215} \times \sqrt[20]{412}$. *Ans.* 1·159.
8. Of $\sqrt{(2\cdot5)^3} \times \sqrt[3]{(5\cdot2)^2}$. [illegible]*ns.* [illegible]·186.
9. Of $\sqrt{\frac{5}{9}} \times \sqrt[3]{\frac{4}{5}}$. [illegible]*ns.* ·69[illegible].

## To Adapt an Expression to Logarithmic Computation.

Let it be required to find the value of $x$ in the equation,

$$\sqrt{x} = \frac{a^2b^3\sqrt[4]{c}}{d^5\sqrt[6]{h}}.$$

We shall have

$$\tfrac{1}{2}\text{Log.}\,x = (2\,\text{Log.}\,a + 3\,\text{Log.}\,b + \tfrac{1}{4}\text{Log.}\,c) - (5\,\text{Log.}\,d + \tfrac{1}{6}\text{Log.}\,h)$$

Or,

$$\text{Log.}\,x = 2(2\,\text{Log.}\,a + 3\,\text{Log.}\,b + \tfrac{1}{4}\text{Log.}\,c - 5\,\text{Log.}\,d - \tfrac{1}{6}\text{Log.}\,h)$$

### Examples.

Adapt to Logarithmic Computation and find the value of $x$ in the following equations:

1. $x = abc \;:\; x = \dfrac{a}{b} \;:\; x = a^2b^3 \;:\; x = \sqrt{a} \times \sqrt[4]{b} \times \sqrt[5]{c}.$

*Ans.* $\text{Log.}\,x = \text{Log.}\,a + \text{Log.}\,b + \text{Log.}\,c : \text{Log.}\,x = \text{Log.}\,a - \text{Log.}\,b :$

$$\text{Log.}\,x = 2\text{Log.}\,a + 3\text{Log.}\,b :$$

$$\text{Log.}\,x = \tfrac{1}{2}\text{Log.}\,a + \tfrac{1}{4}\text{Log.}\,b + \tfrac{1}{5}\text{Log.}\,c.$$

2. $x = \dfrac{\sqrt{a}\,.\,b\,.\,\sqrt[4]{c}}{d} \;:\; x = \dfrac{ab^2\sqrt[4]{c}}{d^3\sqrt{e}} \;:\; x = \dfrac{a^2bc^4\sqrt[3]{d}}{e\,.\,\sqrt{f}\,.\,g^2}.$

*Ans.* $\text{Log.}\,x = \tfrac{1}{2}\text{Log.}\,a + \text{Log.}\,b + \tfrac{1}{4}\text{Log.}\,c - \text{Log.}\,d :$

$$\text{Log.}\,x = \text{Log.}\,a + 2\text{Log.}\,b + \tfrac{1}{4}\text{Log.}\,c - 3\text{Log.}\,d - \tfrac{1}{2}\text{Log.}\,e :$$

$$\text{Log.}\,x = 2\text{Log.}\,a + \text{Log.}\,b + 4\text{Log.}\,c + \tfrac{1}{3}\text{Log.}\,d - \text{Log.}\,e - \tfrac{1}{2}\text{Log.}\,f - 2\text{Log.}\,g.$$

3. $x = \sqrt{\dfrac{ab^2}{c\sqrt{d}}}$ : $x^{\frac{1}{3}} = \sqrt[5]{\dfrac{a^2\sqrt{b}\,.\,\sqrt[3]{c}}{b^2d^4\sqrt[5]{e}}}$ : $a^2 = \dfrac{b\sqrt{d}}{x\,.\,\sqrt[3]{c}}$.

*Ans.* $\text{Log.}\,x = \frac{1}{2}\{\text{Log.}\,a + 2\text{Log.}\,b - \text{Log.}\,c - \frac{1}{2}\text{Log.}\,d\}$:

$$\text{Log.}\,x = \tfrac{3}{5}\{2\text{Log.}\,a + \tfrac{1}{2}\text{Log.}\,b + \tfrac{1}{3}\text{Log.}\,c - 2\text{Log.}\,b - 4\text{Log.}\,d - \tfrac{1}{5}\text{Log.}\,e\}:$$

$$\text{Log.}\,x = \text{Log.}\,b + \tfrac{1}{2}\text{Log.}\,d - 2\text{Log.}\,a - \tfrac{1}{3}\text{Log.}\,c.$$

Find the value of $x$ in the following expressions:

(1) $x = \dfrac{\cdot 0056}{\cdot 000084}$. *Ans.* 66·66.

(2) $x = \dfrac{2247}{1017} \div \dfrac{903}{1107} \times \dfrac{774}{615} \div \dfrac{1926}{565}$. *Ans.* 1.

(3) $x = (7\cdot 54356)^9$. *Ans.* 79101355.

(4) $x = \sqrt[4]{\cdot 248}$. *Ans.* ·7056.

(5) $x^3 = \frac{3}{4}$. *Ans.* ·9085.

(6) $x^{\frac{1}{3}} = \frac{2}{5}$. *Ans.* ·064.

(7) $x^{\frac{5}{9}} = \dfrac{\cdot 42}{\cdot 0059}$. *Ans.* 2159.

(8) $3^x = 5$. *Ans.* 1·46.

(9) $5^x \times 6^{2x} = 1456\cdot 4$. *Ans.* 1·402.

(10) $2^{3^x} = 4$. *Ans.* ·6309.

(11) $\dfrac{a^x}{b^x} = c$. *Ans.* $x = \dfrac{\text{Log.}\,c}{\text{Log.}\,a - \text{Log.}\,b}$.

## Calculation of Expressions.

1. Calculate the value of $\sqrt[3]{\dfrac{1856 \times 31^5 \times (7{\cdot}81)^{10}}{612^2 \times 1928^3}}$

| *Numerator.* | | *Denominator.* | |
|---|---|---|---|
| Log. 1856 = | 3·268578 | 2 Log. 612 = | 5·573502 |
| 5 Log. 31 = | 7·456810 | 3 Log. 1928 = | 9·855321 |
| 10 Log. 7·81 = | 8·926510 | | 15·428823 |

$$\begin{array}{r} 19{\cdot}651898 \\ 15{\cdot}428823 \\ \hline 3\,\big)\,4{\cdot}223075 \\ \hline 1{\cdot}407691 \end{array}$$

*Ans.* 255·6.

2. Find the value of $\dfrac{281 \times 2{\cdot}71828 \times {\cdot}09}{84000 \times {\cdot}7301 \times {\cdot}0073}$.

*Ans.* ·153553.

3. Calculate the value of $\dfrac{8^5 \times 21^2 \times 56^3}{48^3 \times 112^4}$.

*Ans.* ·145833.

4. Calculate the value of $\dfrac{(111)^{\frac{1}{3}} \times (8563)^{\frac{1}{3}} \times (562)^{\frac{1}{2}}}{(98732)^{\frac{1}{7}} \times (3462)^{\frac{1}{4}}}$.

*Ans.* 58·77.

5. Find the value of $\dfrac{4^{-5} \times 5^{-\frac{1}{3}}}{35 \times \sqrt{2}}$.

*Ans.* ·0000115.

6. Find the value of $\dfrac{7\sqrt{15}}{{\cdot}015} \times {\cdot}0139\sqrt{\dfrac{2}{{\cdot}11}}$.

*Ans.* 107·124.

8. Find a 4th proportional to 357·109, 5000·8, and ·031.
*Ans.* ·434111.

9. Find a 4th proportional to $\sqrt[3]{3}$, $\sqrt[4]{4}$, $\sqrt[5]{5}$.
*Ans.* 1·353.

Calculate the values of the following expressions:—

1. $\dfrac{121\sqrt[3]{333}}{\left(\frac{22}{17}\right)^5}$. *Ans.* 231·063.

2. $\dfrac{\cdot 08 \times 240}{1\cdot 5 \times (\cdot 7)^{\frac{4}{5}}}$. *Ans.* 17·027.

3. $\frac{3}{4}\sqrt{1\cdot 1}(\cdot 069)^{\frac{2}{5}}$. *Ans.* ·2699.

4. $\sqrt{13} \cdot \dfrac{1}{(\cdot 506)^3}$. *Ans.* 27·83.

5. $59 + \sqrt[4]{\cdot 0888}$. *Ans.* 108·081.

6. $\dfrac{12}{1\cdot 58} \cdot \dfrac{\sqrt[11]{796}}{(2\frac{1}{9})^{\frac{3}{5}}}$. *Ans.* 8·90302.

7. $\frac{1}{4} \cdot \sqrt[3]{\dfrac{148 \times \cdot 07}{(22)^{\frac{2}{5}}}}$. *Ans.* ·4334.

8. $\dfrac{\cdot 065 \times 7^{1\cdot 2}}{(3\frac{1}{18})^{\frac{5}{6}}}$. *Ans.* ·26472.

9. $\dfrac{(1\cdot 3)^8}{(15)^{\frac{7}{10}}} \cdot \sqrt[7]{\dfrac{145\cdot 67}{(\cdot 095)^{2\cdot 6}}}$. *Ans.* 5·984.

10. $\dfrac{\sqrt{10\cdot 01} \times (\cdot 099)^{\frac{2}{5}}}{(1\frac{1}{13})^{1\cdot 1}}$. *Ans.* 1·157.

11. $\dfrac{100^{\frac{2}{3}}}{19} \cdot \dfrac{1}{\sqrt[5]{\cdot 827}} \cdot (1\frac{1}{5})^{1\cdot 3}$. *Ans.* 1·4928.

12. $\dfrac{1}{711}\sqrt[5]{\dfrac{(2\frac{1}{15})^7}{(\frac{1}{15})^{\frac{3}{4}}}}$. *Ans.* ·004071.

Given $\dfrac{95 \,.\, \text{Cos.}^2 64^\circ}{\text{Tan. } 51^\circ \sqrt{x}} = \text{Cosec.}^3 108^\circ$; to find the value of $x$.

Multiplying by $\sqrt{x}$, we have $\dfrac{95 . \text{Cos.}^2 64^\circ}{\text{Tan. } 51^\circ} = \text{Cosec.}^3 108^\circ \,.\, \sqrt{x}$.

$$\therefore \sqrt{x} = \frac{95 \,.\, \text{Cos.}^2 64^\circ}{\text{Tan.} 51^\circ . \text{Cosec.}^3 108^\circ} = 95 . \text{Cos.}^2 64^\circ . \text{Cot.} 51^\circ . \text{Sin.}^3 108.$$

$$\text{Or,} \quad x = (95 \,.\, \text{Cos.}^2 64 \,.\, \text{Cot. } 51^\circ \,.\, \text{Sin.}^3 108^\circ)^2.$$

| | | |
|---|---|---|
| Log. 95 | = | 1·977724 |
| 2 Log. Cos. 64° | = | 19·283684 |
| Log. Cot. 51° | = | 9·908369 |
| 3 Log. Sin. 108° | = | 29·934618 |
| | | 1·104405 |
| | | 2 |
| | | 2·208810 |

Rejecting 60 from the index—*i.e.*, two tens for $\text{Cos.}^2 64^\circ$, one for Tan. 51° and three for $\text{Sin.}^3 108^\circ$, for reasons stated in *Obs.* p. 56.

$$\therefore \quad x = 161·7.$$

Find $x$ in the following expressions:

(1). $\dfrac{9·5 \,.\, \text{Sin.}^2 47^\circ}{\text{Tan. } 52^\circ \sqrt{x}} = \text{Sec.}^3 101^\circ$. *Ans.* ·0007606.

(2). $\dfrac{11 \,.\, \text{Sin.}^3 115^\circ}{x^2} = \dfrac{1·56}{\sqrt{\text{Cosec. } 60^\circ}}$. *Ans.* 2·558.

(3). $\dfrac{111·1 \text{ Sec.}^3 115^\circ}{\sqrt[3]{x}} = \dfrac{15·6}{\sqrt[5]{\text{Tan.}^4 46^\circ}}$. *Ans.* 162·048.

(4). $x = \dfrac{a \,.\, \text{Tan. A . Sin. B}}{\text{Sin. C}}$, where $a = 416$ feet,

A = 23° 50′ 15″, B = 54° 28′ 30″, C = 147° 32′ 50″.

*Ans.* 278·7 feet.

**Having given the numerical value of a trigonometrical ratio to find the angle.**

**Ex.** Given Sin. A $= \frac{125}{327}$; find angle A.

We have, Log. Sin. A $-$ 10 $=$ Log. 125 $-$ Log. 327.

Or, Log. Sin. A $=$ 10 $+$ Log. 125 $-$ Log. 327.

## Calculation.

$$\begin{array}{lcr} 10+\text{Log. } 125 & = & 12\cdot096910 \\ -\text{Log. } 327 & = & 2\cdot514548 \\ \hline \text{Log. Sin. A} & = & 9\cdot582362 \end{array}$$

$$\therefore \quad A = 22^\circ\ 28'\ 30''.$$

## Examples.

Find angle A in the following examples:

(1) Sin. A $= \frac{23}{45}$. *Ans.* $30^\circ\ 44'\ 15''$.

(2) Cos. A $= \frac{15}{40}$. *Ans.* $67^\circ\ 58'\ 30''$.

(3) Sec. A $= \frac{125}{123}$. *Ans.* $10^\circ\ 15'\ 45''$.

(4) Tan. A $= \frac{123}{345}$. *Ans.* $19^\circ\ 37'\ 15''$.

(5) Cot. A $= \frac{42}{35}$. *Ans.* $39^\circ\ 48'\ 15''$.

(6) Tan.$^2$A $= \frac{49}{36}$. *Ans.* $49^\circ\ 24'\ 0''$.

## SOLUTION OF PLANE TRIANGLES.

*In solving triangles, the student should carefully bear in mind the following principles:—*

1. If the *sides* of a triangle be equal, the *angles* opposite those sides are also equal, and *vice versâ.* (Euc. I., 5, 6.)

2. The angles which one straight line makes with another upon one side of it, are either *two right angles*, or are *together* equal to *two right angles*, or 180°. (Euc. I., 13.)

3. If two straight lines cut one another, the *vertical* or *opposite angles* are *equal.* (Euc. I., 15.)

4. The greater *side* of every triangle is opposite to the greater *angle* and *vice versâ.* (Euc. I., 18, 19.)

5. If a straight line fall upon two *parallel* straight lines, it makes the *alternate* angles equal to one another. (Euc. I., 29.)

6. If a side of a triangle be produced, the *exterior* angle is equal to the *two interior* and opposite angles; and the *three* interior angles are equal to *two* right angles, or 180°. (Euc. I., 32.)*

7. The square upon the *hypotheneuse* of a right-angled triangle, is equal to the squares upon the *sides* containing the right angle.

8. *The angle of elevation* of an object is the angle between the horizontal line, and the line joining the eye of the observer and the object (when the observer is *below* the object).

9. *The angle of depression* of an object is the angle between the horizontal line, and the line joining the eye of the observer and the object (when the observer is *above* the object).

10. The angles of elevation and depression being alternate angles, are therefore equal to one another. (See Appendix.)

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* When the angles are given in compass points the three interior angles are equal to 16 points.

# SOLUTION OF PLANE TRIANGLES.

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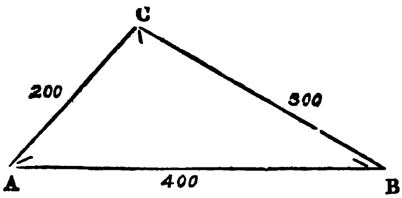
## RULE I.

*Three sides being given, to find an angle.*

### Example.




In triangle ABC, given

$a = 200, \quad b = 300,$

$c = 400$; to find A.

### First Method.

| | | |
|---|---|---|
| Put down the two sides containing A .. .. | 300 .. | Log. 2·477121 |
| | 400 .. | Log. 2·602060 |
| Take their difference.. | 100 | 5·079181 (sum) |
| | (subtract from) | 10· |
| Put down the third side | 200 | |
| | | 4·920819 |
| Take the sum .. | 300 | |
| And difference .. | 100 | |
| And the ½ sum .. | 150 .. | Log. 2·176091 |
| and ½ difference.. | 50 .. | Log. 1·698970 |
| Sum of Logs. .. .. | | 8·795880 = Hav. A. |

$\therefore \quad A = 28^\circ \; 57' \; 15''.$

(continued at page 70.

## Examples for Practice.

1. Given $a = 27$, $b = 32$, $c = 9$; find angle C.

*Ans.* $C = 14^\circ\ 37'\ 30''$.

2. Given $a = 1500$, $b = 1342$, $c = 1110$; find the value of C.

*Ans.* $C = 45^\circ\ 33'\ 30''$.

3. Given $a = 649$, $b = 586$, $c = 757$; find A, B, and C.

*Ans.* $A = 56^\circ\ 4'$; $B = 48^\circ\ 31'$; $C = 75^\circ\ 25'$.

4. Given $a = 50$, $b = 28\frac{1}{2}$, $c = 36\frac{9}{100}$; find the angle A.

*Ans.* $100^\circ\ 47'\ 30''$.

5. Given $a = \cdot 004$, $b = \cdot 0059$, $c = \cdot 0084$; find angle B.

*Ans.* $39^\circ\ 38'\ 15''$.

6. Given $a = 30$, $b = 40$, $c = 50$; find the angle opposite to the greatest side.

*Ans.* $90^\circ$.

7. Given $a = 627$, $b = 1140$, $c = 718\cdot 9$; find the angle opposite the greatest side.

*Ans.* $115^\circ\ 36^{\circ\prime}\ 30''$.

8. The sides of a triangle are in the proportion of $5\cdot 12$, $6\cdot 27$, and $4\cdot 3$; find the difference between the greatest and least angle.

*Ans.* $40^\circ\ 4'\ 15''$.

(continued at page 71.

## RULE I—continued.

*Three sides being given, to find an angle.*

### Second Method.

$$\text{Cos.}\frac{A}{2} = \sqrt{\frac{s.(s-a)}{bc}}$$

where $s =$ half the sum of the sides.

*Calculation.*

| | | |
|---|---|---|
| $a = 20$ | | |
| $b = 30$ | Log. 45 .. 1·653213 | Log. 30 .. 1·477121 |
| $c = 40$ | Log. 25 .. 1·397940 | Log. 40 .. 1·602060 |
| $2\,\vert\,90$ | Sum .. 3·051153 | Log. $bc$ = 3·079181 |
| $s = 45$ | Log. $bc$ .. 3·079181 | |
| $(s-a) = 25$ | $2\,\vert\,\bar{1}$·971972* | $\frac{A}{2} = 14^\circ\ 28'\ 45''$ |
| | $\bar{1}$·985986 | 2 |
| | +10·† | $A = 28^\circ\ 57'\ 30''$ |
| | $\text{Cos.}\frac{A}{2} =$ 9·985986 | |

---

* When the index of a Log. is *negative* and not divisible by a given number, we must add to it as many negative units as will render it divisible, placing the same number of *positive* units before the decimal part of the Log. Thus, in the above example,

Since $-1 = -2 + 1$, we have

$$\frac{\bar{1}\cdot971972}{2} = \frac{\bar{2}+1\cdot971972}{2} = \bar{1} + \cdot985986 = \bar{1}\cdot985986.$$

† 10 is added here to obtain the *tabular* Log. Cosine. *Vide* Chapter on Logarithms, *Obs.* p. 56.

## Examples for Practice.

9. The sides of a triangle are in the proportion of 8, 6·72, and 2·75; find the difference between the greatest and least angle.

*Ans.* $88^\circ\ 39'\ 45''$.

10. Given $a = \frac{1}{4}$, $b = \cdot 541$, $c = \cdot 674$; find the angle opposite the least side.

*Ans.* $20^\circ\ 11'\ 15''$.

11. Two landmarks are 5 miles apart; a ship is stationed 4 miles from one, and 3 miles from the other; find the angle which they subtend at the ship.

*Ans.* $90^\circ$.

12. Two ships are at anchor 798 yards apart: a boat is stationed 460 yards from one, and 654 yards from the other; what angle will they subtend at the boat?

*Ans.* $89^\circ\ 45'\ 45''$.

13. Two headlands are 5·12 miles apart; a ship finds herself 6·27 miles from the westerly one, and due south of it; if the distance of the ship from the easterly one was 4·3 miles, how did it bear from her?

*Ans.* N. $54^\circ\ 8'\ 15''$ E.

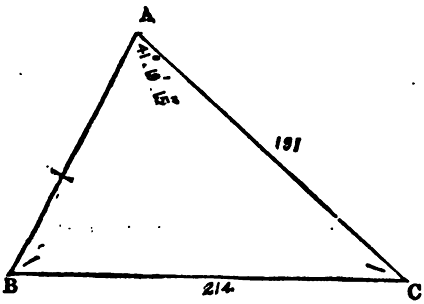
14. There are three islands, A, B and C; the distance between A and B is 20 miles, and C is 30 miles from A and 40 miles from B; A being due North of C, and B to the Eastward of A, how will B bear from C?

*Ans.* N. $28^\circ\ 57'\ 15''$ E.

## RULE II.

***Of two sides and the angles opposite to them, any three being given, to find the fourth.***

### Example.




Given $a=214$, $b=191$, and $A=41^\circ 19' 15''$; to find B, C, and side $c$.

**I When the part required is an angle.**

Write down a proportion having the sine of the *required* angle for the first term, the sine of the *given* angle for the second, and the *sides* respectively opposite to them for the third and fourth terms.

Thus, $\underline{\text{Sin. B}} : \text{Sin. A} :: b : a$

or, $\underline{\text{Sin. B}} : \text{Sin. } 41^\circ\ 19'\ 15'' :: 191 : 214$.

Add together,

$$\left.\begin{array}{ll}\text{Log. Sin. } 41^\circ\ 19'\ 15'' \ldots & 9{\cdot}819725 \\ \text{and, Log. } 191 \ldots\ldots\ldots\ldots & 2{\cdot}281033\end{array}\right\}\ldots\left\{\begin{array}{l}\text{Logs. of the} \\ \text{two \textit{middle}} \\ \text{terms}\end{array}\right\}$$

$$\begin{array}{lll}\text{From the sum} \ldots\ldots\ldots & 12{\cdot}100758 & \\ \text{Take Log. } 214 \ldots & 2{\cdot}330414 & \{\text{Log. of \textit{last} term}\} \\ \hline \text{The remainder} \ldots & 9{\cdot}770344 & = \text{Log. Sin. B.}\end{array}$$

$$\therefore \quad B = 36^\circ\ 6'\ 30''.$$

(continued at page 74.

## Examples for Practice.

1. Given $a = 47$, $b = 53$, $A = 36° 42' 30''$; find B, C, and $c$.

*Ans.* $B = 42° 22' 45''$; $C = 100° 54' 45''$; $c = 77{\cdot}21$.

2. Given $b = 312$, $a = 517$, and $A = 124° 32'$; find B, C, and $c$.

*Ans.* $B = 29° 48' 45''$; $C = 25° 39' 15''$; and $c = 271{\cdot}745$.

3. Given $A = 35° 35' 30''$, $a = 6$, $b = 4$; find the other parts.

*Ans.* $B = 22° 49' 45''$; $C = 121° 34' 45''$; $c = 8{\cdot}783$.

4. Given $A = 40° 15' 35''$; $B = 45°$, $a = {\cdot}025$; find the other parts.

*Ans.* $C = 94° 44' 25''$; $b = {\cdot}02735$; $c = {\cdot}03854$.

5. Given $a = 629$, $b = 765$, and $A = 25° 25' 25''$; find the other parts.

*Ans.* $B = 31° 28' 30''$; $C = 123° 6' 5''$; and $c = 1227{\cdot}32$.

6. There are three ships, A, B, and C, at anchor in a roadstead. At A, the angle subtended by the other two ships, B and C, is found to be $41° 19' 15''$; B is known to be 214 yards from C; and C, 191 yards from A; find the distance between A and B. (See Fig. p. 72.)

*Ans.* 316·3 yards.

7. The distance of the Britannia from the end of Portland Breakwater was 1·5 mile; the distance of the Nothe from the same point being 2 miles, and the angle between these two points being found at the ship to be 45°; it is required to find the distance of the Britannia from the Nothe.

*Ans.* 2·756 miles.

(continued at page 75.

## RULE II—continued.

*Of two sides and the angles opposite to them, any three being given, to find the fourth.*

To find angle C.

| | | | |
|---|---|---|---|
| To angle | A | = | 41° 19′ 15″ |
| Add angle | B | = | 36 6 30 |
| Take the sum (A + B) | | = | 77 25 45 |
| From | | | 180 |
| Then angle | C | = | 102 34 15 |

II. When the required part is a side:

Put the *required* side in the first term, the *given* side in the second, and the angles respectively opposite to them in the third and fourth terms of the proportion.

Thus, $c : a :: \text{Sin. C} : \text{Sin. A}$

| | | | |
|---|---|---|---|
| To | Log. $a$ | = | 2·330414 |
| Add | Log. Sin. C | = | 9·989462 |
| From the sum | | | 12·319876 |
| Take Log. Sin. A | | = | 9·819725 |
| | The remainder | | 2·500151 = Log. $c$. |

$$\therefore \quad c = 316{\cdot}3^*$$

* The side $c$ may also be found as follows:

From the above proportion, we have

$$c = \frac{a \,.\, \text{Sin. C}}{\text{Sin. A}} = a \,.\, \text{Sin. C. Cosec. A}$$

| | | |
|---|---|---|
| Log. $a$ | = | 2·330414 |
| „ Sin. C | = | 9·989462 |
| „ Cosec. A | = | 10·180275 |
| Log. $c$ | = | 2·500151 |

$$\therefore \quad c = 316{\cdot}3.$$

## Examples for Practice.

8. A lighthouse was observed from a ship to bear N. 45° E., and after sailing due South for 6 miles it bore N. 30° E.; find its distance from the ship at the last observation.

*Ans.* 16·39 miles.

9. To determine the distance of a ship at anchor at C I measured a base line AB of 1000 yards on the shore; at A the ship bore N. 57° 50′ E., and at B, N. 6° 41′ W.; required the distance of the ship from A (AB lying E. and W.).

*Ans.* 1100·2 yards.

10. A ship observed a mountain to bear N.E., and after she had sailed W. by S. 20 miles, it bore E.N.E.; required its distance at the second observation.

*Ans.* 29·04 miles.

11. Two iron-clads, A and B, intending to bombard a fort, took up positions 450 yards apart; at A, the angle between B and the fort was 60°, and at B, the angle between A and the fort was 50°; whether was A or B nearer to the fort, and by how much?

*Ans.* A, by 47·9 yards.

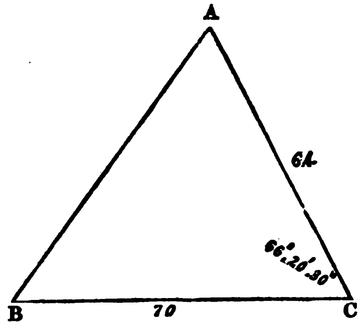
12. A ship anchored near the south-western extremity of a reef which stretches N.E. for a distance of 9 miles, finding the wind coming in strong from E. by N., reaches off to seaward on the port-tack, close-hauled; how far must she sail before she can "go about," so as to clear the other end of the reef, if she can lay within 6 points of the wind?

*Ans.* 7·07 miles.

## RULE III.

***Two sides and the included angle being given, to find the other two angles, and the remaining side.***

### Example.




Given $a = 70$, $b = 64$,

and

angle $C = 66° \, 20' \, 30''$;

to find A, B, and $c$.

First, to find A and B by the proportion,

*Tan. $\frac{1}{2}(A-B)$ : Tan. $\frac{1}{2}(A+B)$ :: $a-b$ : $a+b$.

| | | | |
|---|---|---|---|
| Subtract C or | 66° 20′ 30″ | $a$ = | 70 |
| From | 180 | $b$ = | 64 |
| Then, A+B = | 113 39 30 | $a+b$ = | 134 |
| and $\frac{1}{2}$(A+B) = | 56 49 15 | $a-b$ = | 6 |

∴ Tan $\frac{1}{2}(A-B)$ : Tan. 56° 49′ 45″ :: 6 : 134 (By formula).

Add together—

| | | |
|---|---|---|
| Log. Tan. 56° 49′ 45″ .. | 10·184652 | Logs. of the two *middle* terms |
| and, Log. 6 .. .. .. | ·778151 | |
| From the sum .. | 10·962803 | |
| Take Log. 134 .. | 2·127105 | {Log. of *last* term} |
| The remainder .. | 8·835698 | = Tan. $\frac{1}{2}(A-B)$ |

(continued at page 78.

* When side $b$ is greater than side $a$, angle B is also greater than angle A, and the formula becomes

Tan. $\frac{1}{2}(B-A)$ : Tan. $\frac{1}{2}(B+A)$ :: $b-a$ : $b+a$.

## Examples for Practice.

1. Given $a = 399$, $b = 230$, and $C = 55^\circ\ 2'\ 15''$; find A and B.

*Ans.* $A = 89^\circ\ 45'\ 37''$; $B = 35^\circ\ 12'\ 7''$.

2. Given $b = 700$, $c = 640$, and $A = 66^\circ\ 20'\ 30''$; find B and C.

*Ans.* $B = 60^\circ\ 44'\ 45''$; $C = 52^\circ\ 54'\ 45''$.

3. Given $a = 512$, $b = 627$, $C = 42^\circ\ 53'\ 38''$; find A, B, and side $c$.

*Ans.* $A = 54^\circ\ 8'\ 26''$; $B = 82^\circ\ 57'\ 56''$; $c = 430{\cdot}1$.

4. Given $b = 2\frac{1}{2}$, $c = 2$, $A = 22^\circ\ 20'$; find B, C and $a$.

*Ans.* $B = 108^\circ\ 12'\ 30''$; $C = 49^\circ\ 27'\ 30''$; $a = 1$.

5. Given $a = 798$, $b = 460$, $C = 55^\circ\ 2'\ 15''$; find side $c$.

*Ans.* $c = 654$.

6. Given $c = 48$, $a = 96$, $B = 49^\circ$; find $b$.

*Ans.* $b = 73{\cdot}98$.

7. A ship sails due South 230 miles, then N. 55° E. 399 miles; how far will she be from the place of her departure?

*Ans.* 326·8 miles.

8. A ship sails North 512 miles, then S. 42° 54′ E. 627 miles; what will be her distance from the place left?

*Ans.* 430·1 miles.

(continued at page 79.

## RULE III—continued.

*Two sides and the included angle being given, to find the other two angles, and the remaining side.*

$$\therefore\ \tfrac{1}{2}(A-B) \text{ or, } \tfrac{1}{2}A-\tfrac{1}{2}B = 3^\circ\ 55'\ 0''$$
$$\text{and } \tfrac{1}{2}(A+B) \text{ or, } \tfrac{1}{2}A+\tfrac{1}{2}B = 56\ \ 49\ \ 45$$

$$\therefore \text{ by addition } \ldots\ A = 60\ \ 44\ \ 45$$
$$\text{and, by subtraction} \ldots\ B = 52\ \ 54\ \ 45$$

And $c$ is found as in the preceding Rule, by the formula,

$$c : a :: \text{Sin. } C : \text{Sin. } A.$$

$$\text{Or, } c : 70 :: \text{Sin. } 66^\circ\ 20'\ 30'' : \text{Sin. } 60^\circ\ 44'\ 45''.$$

$$\text{Log. } 70 = 1{\cdot}845098$$
$$\text{Sin. } 66^\circ\ 20'\ 30'' = 9{\cdot}961874$$
$$11{\cdot}806972$$
$$\text{Sin. } 60^\circ\ 44'\ 45'' = 9{\cdot}940746$$
$$\text{Log. } c = 1{\cdot}866226$$
$$\therefore\ c = 73{\cdot}49.$$

## RULE IV.

*When two sides and the included angle are given, to find the third side only.*

**Ex.** Given $a = 70$, $b = 64$, $C = 66^\circ\ 10'\ 30''$; to find $c$.

| | | | |
|---|---|---|---|
| To Log. Tan. $\frac{C}{2}$ | 9·815348 | To Log. Cos. $\frac{C}{2}$ | 9·922748 |
| Add Log. $(a+b)$ | 2·127105 | | |
| From the Sum | 11·942453 | | |
| Take Log. $(a-b)$ | 0·778151 | Add Log. $(a-b)$ | 0·778151 |
| The rem^r is Log. Tan. $\theta$ | 11·164302 | And Log. Sec. $\theta$ | 11·165083 |
| | | Their Sum | 1·865982 = Log. $c$ |
| | | $\therefore\ c = 73{\cdot}44.$ | |

See also Examples in Rule III, from 5 to the end.

## Examples for Practice.

9. Two ships start from the same port; one sails East 798 miles, and the other, N. 35° E. 460 miles; how far will they be apart?

*Ans.* 654 miles.

10. Being in a boat, I find my distance from a ship, A, to be 4·8 cables, and from another, B, to be 9·6 cables (by taking their mast-head angles), and the angle between them to be 49°; find their distance apart.

*Ans.* 7·4 cables.

11. A ship sailing N.W., two islands appeared in sight; one bearing W.N.W., and the other N.; when the ship had sailed 6 miles further, the first bore W. by S., and the other N.E.; required their bearing and distance from each other.

*Ans.* S. 58° 40′ W. 9·71 miles.

12. Two forts protect the mouth of a harbour. I find my distance from one to be 300 yards, and from the other 400 yards, and the angle between them to be 100°. What is their distance apart?

*Ans.* 540 yards.

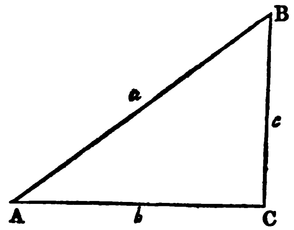
13. Wishing to know the distance of a battery from a town, from which it was not visible on account of a wood intervening, I proceeded to a spot from which both could be seen, and ascertained my distance from the battery to be 800 yards, and from the town 1100 yards, and the angle between them to be 75°; required the distance of the battery from the town.

*Ans.* 1180·9 yards.

# RIGHT-ANGLED TRIANGLES.

## RULE I.

***Of the three sides of a right-angled triangle, any two being given, to find the third.***




By Euclid, Book I, Prop. 47:—

*hypoth.*$^2$ = *base*$^2$ + *perp.*$^2$

or, $a^2 = b^2 + c^2$ (1)

$\therefore \quad c^2 = a^2 - b^2$ (2)

and, $b^2 = a^2 - c^2$ (3)

**Example I.**—Given $b = 40$, $c = 30$; to find $a$.

By Eq$^n$. (1) $a^2 = 40^2 + 30^2 = 1600 + 900 = 2500$.

$$\therefore \quad a = \sqrt{2500} = 50.$$

**Example II.**—Given $a = 50$, $b = 40$; to find $c$.

By (2) $c^2 = a^2 - b^2 = (a - b)(a + b)$

or, $c^2 = 10 \times 90 = 900$

$\therefore \quad c = 30$.

**Example III.**—Given $a = 50$, $c = 30$; to find $b$.

$$b^2 = a^2 - c^2 = (a - c)(a + c)$$
$$b^2 = 20 \times 80 = 1600$$
$$\therefore b = 40.$$

Otherwise, by Logs.—

$b = \sqrt{20 \times 80}$, or, Log. $b = \frac{1}{2}\{\text{Log. } 20 + \text{Log. } 80\}$

$$\begin{array}{rcl} \text{Log. } 20 & = & 1{\cdot}301030 \\ \text{Log. } 80 & = & 1{\cdot}903090 \\ & 2 \,| & 3{\cdot}204120 \\ \text{Log. } b & = & 1{\cdot}602060 \end{array}$$

$$\therefore \quad b = 40.$$

## Examples for Practice.

1. Given $c = 300$, $b = 400$; find $a$.
   *Ans.* 500.

2. Given $a = 500$, $b = 400$; find $c$.
   *Ans.* 300.

3. Given $a = 500$, $c = 300$; find $b$.
   *Ans.* 400.

4. Show that the triangle whose sides are 3, 4, and 5, is right-angled?

5. Show that the triangle whose sides are 2, 3, and 4, cannot be right-angled.

6. A ladder 60 feet long just reaches to the top of a house when its foot is 20 feet from the base; find the height of the house.
   *Ans.* 56·56 feet.

7. A ship sails North 100 miles, then East 50 miles; find the distance made good.
   *Ans.* 111·8 miles.

8. Two ships start from the same port at the same time; one sails North 70 miles, and the other West 30 miles; how far will they be apart?
   *Ans.* 76·16 miles.

9. Two ships start, at the same time, from places on the same meridian, and 100 miles apart; one sails in a North-easterly direction for 15 hours, at the rate of 10 knots an hour, and falls in with the other that had sailed due East from her port; at what rate did the second sail?
   *Ans.* 7·4 knots per hour.

10. A ladder 36 feet long may be so placed that it shall reach a window 30·7 feet from the ground, on one side of the street, and by only turning it over, without moving the foot from its place, it will reach another window 18·9 feet high on the other side; what is the breadth of the street?
   *Ans.* 49·44 feet.

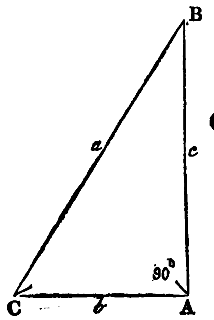
## RULE II.

*When one side and an angle, or two sides are given, to find the other parts of the triangle.*

**Example I.**—Given $A = 90°$, $b$, and C, to find the other parts.

(1) *To find c.*

Put down the side *required* .. .. .. $c$
Under it put the *given* side .. .. .. $b$
Then write down the ratio of the *given* angle formed by the above expression .. } .. Tan C




$$\text{Thus} \quad \frac{c}{b} = \text{Tan. C.}$$

$$\text{and} \quad c = b\,.\,\text{Tan. C.}$$

Or, in Logs.,

$$\text{Log. } c = \text{Log. } b + \text{Log.}^{*}\text{ Tan. C} - 10. \text{ (i)}$$

Similarly, to find $a$.

$$\frac{a}{b} = \text{Sec. C}$$

$$\therefore \; a = b\,.\,\text{Sec. C}$$

$$\therefore \; \text{Log. } a = \text{Log. } b + \text{Log.}^{*}\text{ Sec.} - 10 \text{ (ii)}$$

$$\text{and} \quad B = 90° - C.$$

**Example II.**—Given $A = 90°$, $c = 42$, and $C = 50°\ 10'$; required the other parts.

$$\text{Here, } \frac{b}{c} = \text{Cot. C, or, } b = c\,.\,\text{Cot. C} \qquad (1)$$

$$\frac{a}{c} = \text{Cosec. C, or, } a = c\,.\,\text{Cosec. C.} \qquad (2)$$

(continued at page 84.

* See *Obs.* page 56.

## Examples for Practice.

1. Given $a = 172$, and $A = 23°$, $C = 90°$; find B, $b$, and $c$.
*Ans.* $B = 67°$; $b = 405{\cdot}2$; $c = 440{\cdot}2$.

2. Given $a = 315$, $B = 60°$, $C = 90°$; find A, $b$, and $c$.
*Ans.* $A = 30°$; $b = 545{\cdot}6$; $c = 630$.

3. Given $a = 5{\cdot}1303$, $c = 15$; and $C = 90°$; find $b$, A, and B.
*Ans.* $b = 14{\cdot}09$; $A = 20°$; $B = 70°$.

4. Given $A = 49° \; 14'$, $c = 331$, $B = 90°$; find C, $a$, and $b$.
*Ans.* $C = 40° \; 46'$; $a = 384$; $b = 506\;8$.

5. Given $a = 694{\cdot}73$, $A = 4° \; 44'$, $B = 90°$; find C, $b$, and $c$.
*Ans.* $C = 85° \; 16'$; $b = 8419$; $c = 8390$.

6. Given $c = {\cdot}04$, $C = 40°$, $B = 90°$; find $a$, $b$, and A.
*Ans.* $A = 50°$; $a = {\cdot}04767$; $b = {\cdot}06223$.

7. From the bottom of a tower, BC, a distance, AB equal to 50 yards, is measured on a horizontal plane, and at A the angle BAC is found to be $25° \; 17'$; required the height of the tower BC.
*Ans.* 23·6 yards.

8. A lighthouse, whose height is 95 feet above the level of the sea, is observed from a ship to have an elevation of $3° \; 10' \; 30''$; what is the distance of the ship from the lighthouse?
*Ans.* 570·87 yards.

9. A headland was seen to bear from a ship N.W., and after sailing S.W. 5 miles, it bore North; what was the distance from the headland at the last observation?
*Ans.* 7·07 miles.

(continued at page 85.

## RULE II—continued.

| (1) | | (2) | |
|---|---|---|---|
| Log. $c$ | 1·623249 | Log. $c$ | 1·623249 |
| Cot. C | 9·921247 | Cosec. C | 10·114689 |
| Log. $b$ | *1·544496 | Log. $a$ | *1·737938 |
| ∴ $b$ = | 35·03 | $a$ = | 54·7 |

And B $= 90° - C = 90° - 50° \ 10' = 39° \ 50$.

**Example III.**—Given $A = 90°$, $b = 80$, $c = 150$; to find C.

Here, Tan. $C = \frac{c}{b}$ : or Log. Tan. C − 10 = Log. $c$ − Log. $b$.

*i.e.* Log. Tan. C = 10 + Log. $c$ − Log. $b$

| | | |
|---|---|---|
| 10 + Log. C | = | 12·176091 |
| Log. $b$ | = | 1·903090 |
| Log. Tan. C | = | 10·273001 |

∴ $C = 61° \ 55' \ 45''$.

**Example IV.**—Given $A = 90°$, $a = ·0025$, $B = 30° \ 10'$; find $b$ and $c$.

| (1) | | (2) | |
|---|---|---|---|
| $b = a$ . Sin. B | | $c = a$ . Cos. B | |
| Log: $a$ | $\bar{3}$·397940 | Log. $a$ | $\bar{3}$·397940 |
| Sin. B | 9·701151 | Cos. B | 9·936799 |
| Sum | 7·099091 | Sum | 7·334739 |
| Reject | 10 | Reject | 10 |
| Log. $b$ | $\bar{3}$·099091 | Log. $c$ | $\bar{3}$·334739 |
| ∴ $b$ = | ·001256 | and $c$ = | ·002161 |

* For the reason for rejecting 10, see (i) and (ii), page 82. also Chapter on Logarithms, *Obs.* page 56.

## Examples for Practice.

10. From the maintop of a two-decker, which was 80 feet above the sea, the angle of depression* of a target was 20°; required the distance of the target from the ship.

*Ans.* 219·8 feet.

11. Being ordered to lay down a target at 800 yards from a ship, what angle must I put on my sextant, the height of the main-truck above the water-line being 200 feet?

*Ans.* 4° 45′ 45″.

12. A tower 80 feet high stands on one bank of a river; the angle of elevation of its top, taken from a point exactly opposite on the other bank, is 20°; find the breadth of the river.

*Ans.* 219·8 feet.

13. What is the sun's altitude when an upright post, 20 feet high, casts a shadow 16 feet long?

*Ans.* 51° 20′ 30″.

14. A ship is bound to a port lying 400 miles to the southward and 500 miles to the westward of her; what will be her course, and how far must she sail to reach her port?

*Ans.* S. 51° 20′ 30″ W. 640·3 miles.

15. The angle of depression of a ship at anchor 1,000 yards from the base of a cliff, is found to be 3°; find the height of the cliff.

*Ans.* 52·4 yards.

16. The elevation of a church tower, taken at a distance of 100 feet from its base, was 50°, and the elevation of the spire was 62°; find the height of the tower and spire.

*Ans.* 188·07 feet and 68·9 feet.

---

* For Angles of Depression, see Appendix.

# AREAS OF TRIANGLES.

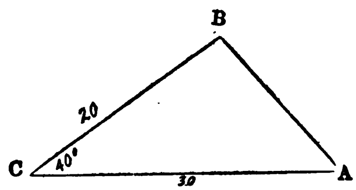
## RULE I.

*Given two sides and the included angle, to find the area.*

The area equals half the product of the two sides multiplied by the sine of the included angle.

### Example.

Given $a = 20$, $b = 30$, and $C = 40°$; to find the area.




$$\text{Area} = \frac{\overset{10}{\cancel{20}} \times 30 \times \text{Sin. } 40°}{\cancel{2}}$$

| | |
|---|---|
| Log. 10 ...... | 1·000000 |
| Log. 30 ...... | 1·477121 |
| Log. Sin. 40°.. | 9·808067 |
| Log. Area .... | 2·285188 |

$\therefore$ Area = 192·8.

*Obs.* When the sides are given in feet, the area will be in *square* feet, when in yards, *square* yards, &c.

## Examples for Practice.

1. Given $b = 35$ feet, $c = 117$ feet, $A = 27°$; find the area.

*Ans.* 929·5 square feet.

2. Given $b = 28·71$ feet, $c = 103·22$ feet, $A = 30°$; find the area.

*Ans.* 740·86 square feet.

3. Given $b = 123·2$ feet, $c = 76$ feet, $A = 49°$; find the area.

*Ans.* 3533·2 square feet.

4. Given $b = 1000$ yards, $c = 2\frac{1}{2}$ miles, $A = 42°$; calculate the area in square miles.

*Ans.* ·4752 square miles.

5. Given $b = 127·3$, $c = 892·7$, $A = 126°$; find the area.

*Ans.* 45968·625.

6. The area of a triangle is 192·8 square feet, and two of its sides are 20 and 30 feet respectively; find the angle between them.

*Ans.* 40°.

7. The area of a triangle is 929·5 square yards, and two of its sides are 35 and 117 yards respectively; find the angle contained by them.

*Ans.* 27°.

8. Given the area of a triangle 3533·2 square feet, one angle 49°, and one of the sides including that angle 76 feet; find the other side.

*Ans.* 123·2 feet.

9. If one angle of a triangle be 30°, and one of the sides containing it be 28·71 feet, what must be the length of the other, in order that the triangle may contain 740·86 square feet?

*Ans.* 103·22 feet.

## RULE II.

*Given three sides, to find the area.*

$$\text{Area} = \sqrt{s \cdot \overline{s-a} \cdot \overline{s-b} \cdot \overline{s-c}},$$

Where $s$ is equal to half the sum of the sides.

### Example.

Given $a = 17{\cdot}28$, $b = 13{\cdot}2$, $c = 14{\cdot}62$; to find the area.

| | | | |
|---|---|---|---|
| $a = 17{\cdot}28$ | $s = 22{\cdot}55$ | $s = 22{\cdot}55$ | $s = 22{\cdot}55$ |
| $b = 13{\cdot}20$ | $a = 17{\cdot}28$ | $b = 13{\cdot}20$ | $c = 14{\cdot}62$ |
| $c = 14{\cdot}62$ | $s-a=5{\cdot}27$ | $s-b=9{\cdot}35$ | $s-c=7{\cdot}93$ |
| $2\,\underline{\vert\, 45{\cdot}10}$ | | | |
| $s = 22{\cdot}55$ | | | |

$$\begin{aligned}
\text{Log. } s &= 1{\cdot}353146 \\
\text{Log. } (s-a) &= 0{\cdot}721810 \\
\text{Log. } (s-b) &= 0{\cdot}970811 \\
\text{Log. } (s-c) &= 0{\cdot}899273 \\
2\,\vert\, & \overline{3{\cdot}945040} \\
& 1{\cdot}972520
\end{aligned}$$

$\therefore$ Area $= 93{\cdot}86$.

*Obs.* If the sides are given in feet the area will be in *square* feet, if in yards, *square* yards, &c.

## Examples for Practice.

1. Given $a = 131$, $b = 246$, $c = 327$; find the area.
*Ans.* 14357·8 square units.

2. Given $a = 2·05$ yards, $b = 1·67$ yards, $c = 2·7$ yards; find the area.
*Ans.* 1·7101 yards.

3. Given $a = 1800$ feet, $b = 1728$ feet, $c = 1521$ feet; calculate the area in square yards.
*Ans.* $134343·\dot{2}$ square yards.

4. Given $a = 0·23$ feet, $b = ·34$ feet, $c = ·45$ feet; find the area.
*Ans.* ·03816 square feet.

5. Given $a = 507$ miles, $b = 603$ miles, $c = 721$ miles; find the area.
*Ans.* 150768 square miles.

6. If the three sides of a triangular field are 131, 246, 327 yards, respectively, how many acres does it contain?
*Ans.* 2·966 acres.

7. An island is in the form of a triangle, whose sides are 20·5, 16·7, and 27 miles, respectively; what is the area in square miles?
*Ans.* 171·01 square miles.

8. How many yards of canvas 2 feet wide will it take to make a jib whose sides are 6, 10, and 12 yards?
*Ans.* 44·895 yards.

9. Find the cost of paving a triangular court, whose sides are 50, 60, and 70 feet, respectively, with granite at 5*s.* per square yard?
*Ans.* £40 16*s.* 6*d.*

---

NOTE.—To find the area of a trapezium, or of any polygonal figure, divide it into triangles, then the sum of the areas of all these triangles will be the area of the figure.

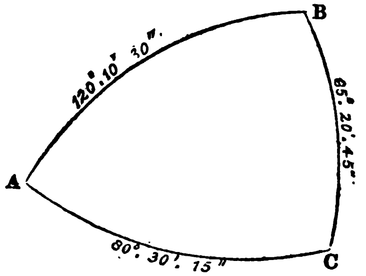
# SOLUTION OF SPHERICAL TRIANGLES.

## RULE I.

*Three sides being given, to find an angle.*

### Example.

Given $a = 65^\circ\ 20'\ 45''$, $b = 80^\circ\ 30'\ 15''$, $c = 120^\circ\ 10'\ 30''$; to find A.




| | | | | Logs. |
|---|---|---|---|---|
| Put down the two sides containing A ...... | 120° 10′ 30″*.. | | Cosec. | 10·063238 |
| | 80 30 15 | .. | Cosec. | 10·005992 |
| Take their difference .. | 39 40 15 | | | |
| Put down the third side | 65 20 45 | | | |
| Take sum .......... | 105 1 0 | .. | ½ Hav. | 4·899515 |
| And difference ..... | 25 40 30 | .. | ½ Hav. | 4·346718 |
| Sum of Logs., rejecting 20 .................. | | | | 9·315463 |
| | | | = Log. Hav. A | |

$\therefore\quad A = 54^\circ\ 5'\ 32''.$

* When the Log. Cosecant of an angle greater than 90° is required, diminish the given angle by 90°, and take the Log. Secant of the remainder,

Thus Log. Cosecant 120° 10′ 30″ = Log. Sec. 30° 10′ 30″.

This rule also applies to the other Trigonometrical Ratios of an angle greater than 90°. Thus, for Log. Cos. 120° take Log. Sin. 30°, for Log. Cot. 120° take Log. Tan. 30°, &c.

## Examples for Practice.

1. Given $a = 120° 58'$, $b = 105° 6'$, and $c = 108° 41' 30''$; find A, B, and C.

*Ans.* $A = 130° 56'$; $B = 121° 35'$; $C = 123° 18'$.

2. Given $a = 49° 10'$, $b = 58° 25'$, $c = 56° 42'$; find A, B, and C.

*Ans.* $A = 59° 2'$; $B = 74° 54'$; $C = 71° 18' 30''$.

3. Given $a = 119° 42' 20''$, $b = 108° 4' 18''$, $c = 68° 53' 42''$; find A, B, and C.

*Ans.* $A = 115° 39'$; $B = 99° 21' 30''$; $C = 75° 31' 45''$.

4. Given $a = 64° 21' 15''$, $b = 80° 38' 45''$, $c = 104° 28' 30''$; find A, B, and C.

*Ans.* $A = 60° 17' 15''$; $B = 71° 56'$; $C = 111° 6' 15''$.

5. Given $a = 87° 10' 15''$, $b = 62° 36' 45''$, $c = 100° 10' 15''$; find A, B, C.

*Ans.* $A = 81° 24' 15''$; $B = 61° 31' 30''$; $C = 102° 59'$.

6. Three points, A, B, and C, are taken on the surface of a globe; the arc of the great circle joining A and B is 124° 10′, those adjoining B and C, and C and A, are 89° 0′ 15″, and 108° 40′, respectively; find the angle subtended by the arc AB.

*Ans.* 125° 56′ 15″.

7. The Colatitude of Liverpool is 36° 35′; that of New York, 49° 18′; and the arc of the great circle joining the two places is 47° 52′; find their difference of longitude.

*Ans.* 70° 59′.

8. The Colatitude of a place is 39° 12′; the Sun's polar distance is 66° 32′ 15″; and his zenith distance 43° 40′; find the polar angle.

*Ans.* 44° 19′, or, $2^h\ 57^m\ 16^s$.

9. The Latitude of a place is 50° 48′ N., the Sun's alt. 46° 20′ (West of Meridian), and his declination 23° 27′ 45″ N., find the azimuth.

*Ans.* N. 111° 51′ W.

10. In Latitude 39° 11′ N., the altitude of the Sun (West of the Meridian), was 16° 59′, and his declination 15° 25′ S. (or, PD 105° 25′); find the apparent time.

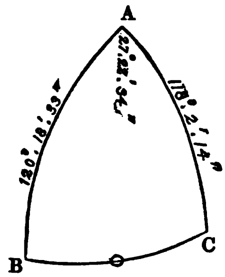
*Ans.* $3^h\ 28^m\ 0^s$.

## RULE II.

*Two sides and the included angle being given, to find the third side.*

### Example.

Given $b = 118° \ 2' \ 14''$; $c = 120' \ 18' \ 33''$; $A = 27° \ 22' \ 34''$; to find $a$.




| | | | | |
|---|---|---|---|---|
| Angle A | = | 27° 22′ 34″ | .... | Hav. 8·748126 |
| Side $b$ | = | 118 2 14 | .... | Sin. 9·945784 |
| Side $c$ | = | 120 18 33 | .... | Sin. 9·936173 |
| Difference | = | 2 16 19 | | 8·630083 = Hav. $\theta$ |

$\therefore \ \theta = 23° \ 50' \ 30''.$

| | | |
|---|---|---|
| * Vers. 23° 50′ 30″ | ...... | 0085333 |
| Vers. 2° 16′ 19″ | ...... | 0000785 |
| Sum | ...... | 0086118 = Vers. $a$ |

$\therefore \ a = 23° \ 57' \ 9''.$

This Rule may be thus expressed:—

$$\text{Vers. } a = \text{Vers. } (b \backsim c) + \text{Vers. } \theta$$

Where $\theta$ is obtained from the formula

$$\text{Hav. } \theta = \text{Sin. } b \,.\, \text{Sin. } c \,.\, \text{Hav. A.}$$

---

* For an explanation of the manner of using the Table of Versines, see Arts. VI. and VII., page 53.

## Examples for Practice.

1. Given $a = 49° 10'$, $b = 58° 25'$, and $C = 71° 18' 30''$; find $c$.

*Ans.* $c = 56° 42'$.

2. Given $b = 108° 4' 18''$, $c = 68° 53' 45''$, and $A = 115° 38' 45''$; find $a$.

*Ans.* $a = 119° 42' 17''$.

3. Given $a = 87° 10' 15''$, $c = 100° 10' 15''$, and $B = 61° 31' 15''$; find $b$.

*Ans.* $b = 62° 36' 30''$.

4. Given $A = 96° 32'$, $b = 76° 42'$, $c = 89° 10' 30''$; find $a$.

*Ans.* $a = 96° 10' 2''$.

5. Given $A = 50°$, $b=70° 45' 10''$, and $c=62° 10' 15''$; find $a$.

*Ans.* $a = 46° 19' 37''$.

6. On a globe are taken three points, P, A, B; one of which, P, is at the pole; PA is 87° 10′ 15″, PB 62° 36′ 45″; and the angle included between the great circles drawn from P to A and B, is 102° 58′ 30″; required the distance between A and B.

*Ans.* 100° 9′ 33″.

7. The Colatitude of Liverpool is 36° 35′; of New York, 49° 18′; and their difference of longitude, 71°. Find the distance between the two places on the arc of a great circle.

*Ans.* 2872 miles.

8. Required the distance from Portsmouth to Buenos Ayres.

Lat. of Portsmouth.... 50° 48′ N. .. Long. 1° 6′ W.
Lat. of Buenos Ayres.. 34° 37′ S. .. Long. 58° 24′ W.

*Ans.* 5949·8 miles.

9. Given Lat. of place, 50° 48′ N., Sun's declination, 16° N., and hour angle, 38°; find his zenith distance and altitude.

*Ans.* Z. D. 46° 11′; Alt. 43° 49′.

10. Required the distance of the Moon from $\alpha$ Leonis (*Regulus*); the right ascension and declination of the former being 0° 32′ 45″, and 5° 19′ S., and of the latter, 148° 18′ 45″, and 13° 10′ 15″ N.

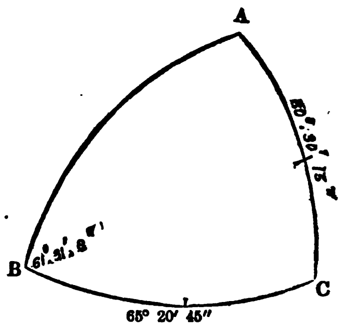
*Ans.* 147° 16′.

## RULE III.

*Of two sides, and the opposite angles, any three being given, to find a fourth.*

### Example I.

Given $a = 65° 20' 45''$, $b = 80° 30' 15''$, $B = 61° 31' 8''$; to find A.




I. *When the part required is an angle.*

Write down a proportion having the sine of the *required* angle in the *first* term, the sine of the *known* angle in the second, and the sines of the sides respectively *opposite* to them in the third and fourth terms:—

Thus,

$\underline{\text{Sin. A}}$ : Sin. B :: Sin. $a$ : Sin. $b$.

| | | |
|---|---|---|
| Add together Log. Sin. B .. | 9·943984 | Logs. of the two *middle* terms |
| And Log. Sin. $a$ .. | 9·958488 | |
| From the sum ............ | 19·902472 | |
| Subtract Log. Sin. $b$........ | 9·994008 | {Log. of *last* term} |
| The remainder ............ | 9·908464 | = Log. Sin. A |

$\therefore$ A = 54° 5′ 31″.

(continued at page 96.

## Examples for Practice.

1. Given $a = 49° 10'$, $b = 58° 25'$, and $B = 74° 54'$; find A.

*Ans.* $A = 59° 2'$.

2. Given $a = 87° 10' 15''$, $b = 62° 36' 45''$, and $B = 61° 31' 15''$; find A.

*Ans.* $A = 81° 24'$.

3. Given $c = 56° 42'$, $b = 58° 25'$, and $C = 71° 18' 30''$; find B.

*Ans.* $B = 74° 54'$.

4. Given $a = 119° 42' 20''$, $b = 108° 4' 18''$, and $B = 99° 21' 15''$; find A.

*Ans.* $A = 115° 38' 45''$.

5. Given $A = 81° 24'$, $a = 87° 10' 15''$, and $c = 100° 10' 15''$; find C.

*Ans.* $C = 102° 59'$.

6. Given $a = 124° 10'$, $b = 89° 0' 15''$, and $A = 125° 56' 30''$; find B.

*Ans.* $B = 78° 3' 15''$.

7. Given $A = 115° 38' 45''$, $B = 99° 21' 15''$, $b = 108° 4' 18''$; find $a$.

*Ans.* $a = 119° 42' 15''$.

8. In question 7, preceding rule, find the Course from Liverpool to New York, at starting.

*Ans.* N. 75° 10′ 30″ W.

(continued at page 97.

## RULE III—continued.

*Of two sides, and the opposite angles, any three being given, to find a fourth.*

### Example II.

Given $a = 65° \ 20' \ 45''$, $c = 120° \ 10' \ 30''$, $A = 94° \ 5' \ 31''$; to find C.

Here, $\underline{\text{Sin. C}} : \text{Sin. A} :: \text{Sin. } c : \text{Sin. } a.$

| | | | |
|---|---|---|---|
| Sin. A = | 9·998892 | Angle = | 71° 35′ |
| Sin. $c$ = | 9·936762 | | 180 0 |
| Sum = | 19·935654 | C = | 108 25 |
| Sin. $a$ = | 9·958488 | | |
| Sin. C = | 9·977166 | | |

In this case the angle must be subtracted from 180° to obtain C, for it is evident that C will be an obtuse angle, because $c$ being greater than $a$, angle C will also be greater than angle A, and therefore greater than 90°, and since there are two angles each less than 180°, having the same sine, we must take that angle which satisfies the conditions of the question.

II. *When the part required is a side,* put the sine of the *required side* in the first term, the sine of the *known* side in the second, and the sines of the angles respectively *opposite* to them, in the third and fourth terms of the proportion, and proceed as before.

## Examples for Practice.

9. Find the bearing of Buenos Ayres from Portsmouth, from the data in question 8, preceding rule.

*Ans.* S. 44° 32′ 45″ W.

10. The azimuth of a heavenly body was N. 111° 51′ W., its altitude at the same time was 46° 20′, and declination 23° 27′ 45″ N.; required the apparent time.

*Ans.* $2^h\ 57^m\ 16^s$.

11. The hour angle of a heavenly body was 44° 19′, its altitude at the same time was 46° 20′ (West of meridian), and declination 23° 27′ 45″ N.; required its azimuth.

*Ans.* N. 111° 51′ W.

12. What was the Sun's altitude at $2^h\ 57^m\ 16^s$, when his declination was 23° 27′ 45″ N., and bearing N. 111° 51′ W.?

*Ans.* 46° 20′.

13. What was the Sun's declination at $2^h\ 57^m\ 16^s$, when his altitude was 46° 20′, and bearing N. 111° 51′ W.?

*Ans.* 23° 27′ 45″ N.

14. At $2^h\ 57^m\ 16^s$, P.M., the Sun bore N. 111° 51′ W., his declination being 23° 27′ 45″ N.; find the length of the shadow cast by a pole 4 feet long, set up vertically in the ground.

*Ans.* 3·818 feet.

15. What is the Sun's altitude at six o'clock, at a place in lat. 50° N., when the declination is 20° N.?

*Ans.* 15° 11′ 30″.

## RULE IV.

*Two sides and the included angle being given, to find the other two angles.*

I. Find the third side by Rule II:
II. Then the other angles may be found by Rule III:

Or, They may be found independently by the formulæ—

$$\text{Tan. } \tfrac{1}{2}(A-B) = \text{Sin. } \tfrac{1}{2}(a-b) \text{ Cosec. } \tfrac{1}{2}(a+b) \text{ Cot. } \frac{C}{2} \qquad (1)$$

$$\text{Tan. } \tfrac{1}{2}(A+B) = \text{Cos. } \tfrac{1}{2}(a-b) \text{ Sec. } \tfrac{1}{2}(a+b) \text{ Cot. } \frac{C}{2} \qquad (2)$$

Which formulæ determine $\frac{1}{2}(A-B)$ and $\frac{1}{2}(A+B)$.

Whence, $A = \frac{1}{2}(A-B) + \frac{1}{2}(A+B)$
And, $B = \frac{1}{2}(A-B) - \frac{1}{2}(A+B)$.

*Obs.* If $\frac{1}{2}(a+b)$ be *greater* than 90°, Sec. $\frac{1}{2}(a+b)$ will be *negative*, therefore the angle resulting from formula (2) must be subtracted from 180°, to obtain $\frac{1}{2}(A+B)$.

## RULE V.

*Given two angles, and the side between them, to find the other two sides.*

Make use of the two formulæ—

$$\text{Tan. } \tfrac{1}{2}(a-b) = \text{Sin. } \tfrac{1}{2}(A-B) \text{ Cosec. } \tfrac{1}{2}(A+B) \,.\, \text{Tan. } \frac{c}{2}. \qquad (1)$$

$$\text{Tan. } \tfrac{1}{2}(a+b) = \text{Cos. } \tfrac{1}{2}(A-B) \text{ Sec. } \tfrac{1}{2}(A+B) \text{ Tan. } \frac{c}{2}. \qquad (2)$$

*Obs.* If $\frac{1}{2}(A+B)$ *exceed* 90°, Sec. $\frac{1}{2}(A+B)$ will be *negative*, therefore the angle resulting from formula (2) must be subtracted from 180° in order to get $\frac{1}{2}(a+b)$.

Whence, $a = \frac{1}{2}(a-b) + \frac{1}{2}(a+b)$
And, $b = \frac{1}{2}(a-b) - \frac{1}{2}(a+b)$.

## Examples for Practice.

1. Given $a = 54° 10'$, $b = 39° 0' 15''$, and $C = 108° 40'$; find A and B.

*Ans.* $A = 53° 24' 15''$; $B = 38° 33' 45''$.

2. Given $b = 58° 25'$, $c = 56° 42'$, and $A = 59° 2'$; find B and C.

*Ans.* $B = 74° 54'$; $C = 71° 18' 30''$.

3. Given $a = 87° 10' 15''$, $b = 62° 36'$, and $C = 102° 59'$; find A and B.

*Ans.* $A = 81° 24'$; $B = 61° 31' 15''$.

4. Given $b = 62° 36' 45''$, $c = 100° 10' 15''$, and $A = 81° 24'$; find C and B.

*Ans.* $C = 102° 59'$; $B = 61° 31' 15''$.

5. In latitude 50° 48′ N., when the Sun's declination was 12° 29′ N., and hour angle $2^h\ 53^m\ 1^s$; required the azimuth.

*Ans.* N. 121° 47′ E.

6. In latitude 50° 48′ N., when the Sun's declination was 23° 27′ N., and hour angle $1^h\ 3^m\ 23^s$; what was its azimuth?

*Ans.* N. 149° 55′ 15″ E.

## Examples in Rule V.

1. Given $A = 59° 2'$, $B = 74° 54'$, and $c = 56° 42'$; find sides $a$ and $b$.

*Ans.* $b = 58° 25'$; $a = 49° 10'$.

2. Given $A = 115° 38' 45''$, $B = 99° 21' 15''$, and $c = 68° 53' 42''$; find sides $a$ and $b$.

*Ans.* $a = 119° 42' 20''$; $b = 108° 4' 18''$.

3. Given $A = 81° 24'$, $C = 102° 59'$, and $b = 62° 36' 45''$; find sides $a$ and $c$.

*Ans.* $a = 87° 10' 15''$; $c = 100° 10' 15''$.

4. At a place in latitude 33° 18′ N., the Sun's hour angle was 59° 2′, when its azimuth was N. 74° 54′ E.; find its N. P. D. (North polar distance.)

*Ans.* 58° 25′.

## RIGHT-ANGLED TRIANGLES.

---

DEFINITIONS of *circular* parts, *middle* part, *adjacent*, and *opposite* parts.

In a right-angled triangle (the right angle being left out of the question) the two sides adjoining the right angle and the *complements* of the hypothenuse, and of the other two angles, are called the five *circular* parts.

Any one of these circular parts may be called a *middle* part.

### To find the Middle Part.

The *middle* part is that side or angle which touches *both* the other parts, or does not touch either of them.

When the three parts concerned in the question are so situated with respect to each other, that they *all* touch, then the one *between* the two others is called the middle part, and the two others are called *adjacent* (or adjoining) parts.

When the three parts concerned in the question are so situated with respect to each other that one of them is *entirely separated* from the other two, it is called the middle part, and the other two are called *opposite* parts.

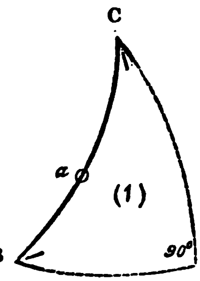
(NOTE I.)—In the solution of right-angled spherical triangles there must always be two parts given, to find the part required.

(NOTE 2.)—The right angle is not taken into consideration, but is left out of the question; and the two sides including it are supposed to touch one another.

(NOTE 3.)—The middle part may be either one of the given parts, or may be a required part.

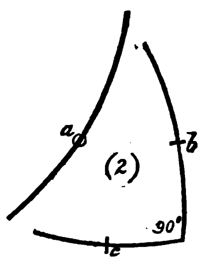
The middle part may be easily found by attention to the following remarks:—

In *Fig.* 1. Suppose that the angles B and C are given, to find side *a*.



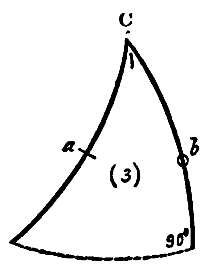

Here it will be observed that the side *a touches* angle B and angle C, and being situated between B and C, is called the middle part. Angles B and C are in this case called adjacent parts.

In Fig. 2. Suppose that the sides *b* and *c* are given, to find side *a*.

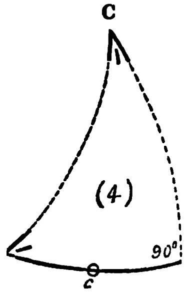



It will here be noticed that side *a* is entirely separated from sides *b* and *c*, by the spaces intervening at the angles of the triangle; *a* is therefore the middle part, and *b* and *c* are the *opposite* parts.

In *Fig.* 3. Suppose that the side *a* and the angle C are given, to find side *b*.

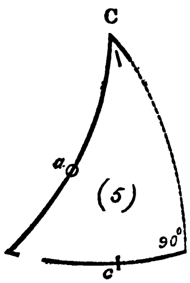



Here it will be seen that angle C touches side *a* and side *b*, and lying between them, is called the middle part; *a* and *b* in this case being *adjacent* parts.



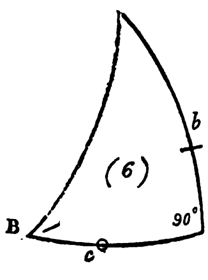

In *Fig.* 4. Suppose that angles C and B are given, to find side *c*.

In this case angle C is entirely separated from the two other parts, and is therefore the middle part; ∠B and side *c* being the *opposite* parts.




In *Fig.* 5. Suppose that angle C and side *c* are given, to find side *a*.

Here it will be seen that side *c* is entirely separated from angle C and side *a*; *c* is therefore the middle part, and C and *a* are *opposite* parts.




In *Fig.* 6. Suppose that angle B and side *b* are given, to find side *c*.

Here *c* touches angle B and side *b*, and, being situated between them, is the middle part; B and *b* being the adjacent parts.

Napier's Rules for the solution of right-angled triangles are these:—

I. Sine of middle part = product of tangents of adjacent parts.

II. Sine of middle part = product of cosines of opposite parts.

Which may easily be remembered thus:—

Sin. Mid. = Tan. Ad.
Sin. Mid. = Cos. Opp.

By these rules, when any two parts are given, a third may be found.

But in applying them we must be careful to take the *complements* of the *hypothenuse* and of the two angles *adjacent* to it.

Then, having written down the equation that embraces the two given parts and the part required, put a dash under the latter, and determine whether it is positive or negative by applying the proper signs (+ or —) to each term by the following rule:—

*When an angle is greater than* 90°, *all its trigonometrical ratios employed in this rule, except the* sine *and* cosecant, *are* negative.

If the sign of the required part is positive the angle given by the formula is the one sought, but if *negative* this angle must be subtracted from 180°.

## Example.

Given $B = 120°$, $C = 60°$, to determine whether $a$ is positive or negative in the following equation:—

$$\underline{\text{Cos. } a} = \overset{-}{\text{Cot. B}} \;.\; \overset{+}{\text{Cot. C}}$$

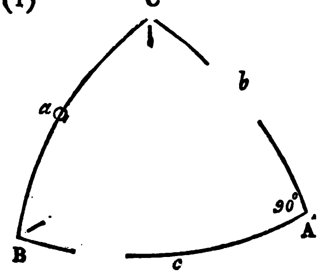
Here, since the signs on the right-hand side of the equation produce minus, Cos. $a$ will be negative, in which case the resulting angle must be subtracted from 180° to obtain $a$, because Cos. $(180° - a) = -$Cos. $a$ (Chap. I).

## Example.

In the right-angled triangle ABC, A being the right angle, let $B = 79° \ 19'$, and $C = 134° \ 32' \ 45''$, be given to determine the other parts, $a$, $b$, $c$.

(1) *To find a, Fig.* 1.

(1)




It is evident that, if three parts, $a$, B, C, be connected together, $a$ will be the ***middle*** part, and B and C the adjacent parts.

∴ By Rule I., Sin. Co. $a$ = Tan. Co. B. Tan. Co. C*

$$\text{or,}\quad \overset{-}{\text{Cos. } a} = \overset{+}{\text{Cot. B}} \,.\, \overset{-}{\text{Cot. C}} \qquad (1)$$

Cos. $a$ will be negative; therefore the resulting angle must be subtracted from 180° to obtain $a$, because $-$Cos. $a$ = Cos. $(180° - a)$. (See page 106.)

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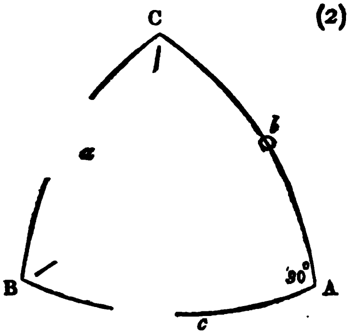
* The Sine complement is equal to the Cosine, the Tan. complement to the Cotan., &c. (See Chap. I., Art. 8.)

(2) *To find* b, *Fig.* 2.

(2)




If B be taken as *middle* part, C and *b*, being separated from it, will be *opposite* parts.

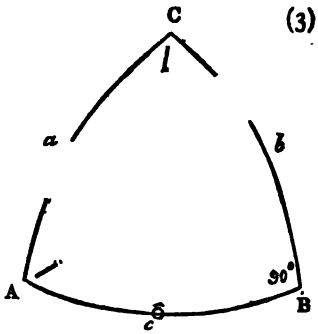
$$\therefore \quad \text{By Rule II., Sin. Co. B} = \text{Cos. Co. C . Cos. } b$$

$$\text{or,} \quad \overset{+}{\text{Cos.}}\ \text{B} = \overset{+}{\text{Sin. C}}\ .\ \underline{\overset{+}{\text{Cos.}}\ b}$$

$$\therefore \quad \text{Cos. } b = \frac{\text{Cos. B}}{\text{Sin. C}} = \text{Cos. B . Cosec. C.} \qquad (2)$$

(3) *To find* c, *Fig.* 3.

(3)




Here it will be seen that two of the parts concerned, B and *c*, are *separated* from C, the third part, which accordingly is taken as a middle part, therefore, by Rule II.,

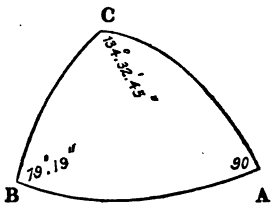
$$\text{Sin. Co. C} = \text{Cos. Co. B . Cos. } c$$

$$\text{or,} \quad \overset{-}{\text{Cos.}}\ \text{C} = \overset{+}{\text{Sin.}}\ \text{B}\ .\ \underline{\overset{-}{\text{Cos.}}\ c}$$

$$\therefore \quad \overset{-}{\text{Cos.}}\ c = \frac{\text{Cos. C}}{\text{Sin. B}} = \text{Cos. C . Cosec. B.} \qquad (3)$$

And Cos. *c* will be negative; so that the resulting angle must be subtracted from 180° to obtain *c*.

The working of Example on page 104, in which are given $A = 90°$, $B = 79° 19'$, $C = 134° 32' 45''$; to find the other parts.




By formulæ, pages 104 and 105, we have

$$-\text{Cos.}\,a = \text{Cot. B . Cot. C.} \tag{1}$$

$$\text{Cos.}\,b = \frac{\text{Cos. B}}{\text{Sin. C}} = \text{Cos. B . Cosec. C} \tag{2}$$

$$-\text{Cos.}\,c = \frac{\text{Cos. C}}{\text{Sin. B}} = \text{Cos. C . Cosec. B.} \tag{3}$$

*Calculation.*

| (1) | | (2) | | (3) | |
|---|---|---|---|---|---|
| Log. Cot. B | 9·275658 | Cos. B .. | 9·268065 | Cos. C .. | 9·846015 |
| Cot. C | 9·993115 | Cosec. C .. | 10·147100 | Cosec. B .. | 10·007594 |
| −Cos. $a$ .. | 9·268773 | Cos. $b$ .. | 9·415165 | −Cos. $c$ .. | 9·853609 |
| | 79° 18′ | $b$ = | 74°55′30″ | | 44° 27′ |
| | 180 0 | | | | 180 0 |
| $a$ = | 100 42 | | | $c$ = | 135 33 |

We reject 10 in the index under (1), (2) and (3), because as

$$\text{Cos.}\,a = \text{Cot. B . Cot. C,}$$

∴ In Logs.,

$$\text{Log. Cos.}\,a - 10 = \text{Log. Cot. B} - 10 + \text{Log. Cot. C} - 10.$$

*i.e.* $\text{Log. Cos.}\,a = \text{Log. Cot. B} + \text{Log. Cot. C} - 10.$

And so in the other Cases.

NOTE.—If $a$ and B were given to find $b$, we should have Sin. $b$ = Sin. $a$ . Sin. B. The solution would in this case be ambiguous, since there are *two* angles less than 180° corresponding to a given Sine. But the ambiguity may be removed by this consideration,—that if B be *greater* than 90°, $b$ will also be *greater* than 90°; and if B be *less* than 90°, $b$ will be *less* in like manner, that is, ***an angle and the side opposite to it are of like affection.***

## Examples for Practice.

1. Given $A=90°$, $B=34° 27' 30''$, and $c=46° 18' 23''$; find $a$, $b$, and C.

*Ans.* $a=51° 46' 15''$; $b=26° 23' 15''$; $C=66° 59' 30''$.

2. Given $A=90°$, $B=100°$, $C=87° 10'$; find $a, b, c$.

*Ans.* $a=90° 30'$; $b=100° 0' 45''$; $c=87° 7' 15''$.

3. Given $B=90°$, $A=91° 25' 58''$, $C=53° 15'$; find the sides.

*Ans.* $a=91° 47' 15''$; $b=91° 4' 15''$; $c=53° 14' 20''$.

4. Given $B=90°$, $a=91° 47' 15''$, $c=53° 14' 20''$; find the other parts.

*Ans.* $A=91° 25' 45''$; $C=53° 15'$; $b=91° 4' 15''$.

5. Given $A=23° 40' 12''$, $B=90°$, and $c=118° 21' 4''$; find the other parts.

*Ans.* $a=21° 5' 45''$; $b=116° 18'$; $C=100° 59' 30''$.

6. Given $a=100° 42'$, $B=78° 10'$, $A=90°$; find the angle C.

*Ans.* $C=131° 32' 45''$.

7. Given the Sun's altitude, when due West, 30°, and its declination 20° N.; required the latitude.

*Ans.* 43° 9′ 15″ N.

8. Given the Sun's declination, 23° 27′ 45″ N., and latitude 50° 48′ N.; find his altitude, and the time when he is on the prime vertical.

*Ans.* Alt. 30° 55′; Time $4^h\ 37^m\ 4^s$.

9. Given the Sun's altitude at Six o'clock, 18° 45′, and declination 20° 4′ N.; find the latitude.

*Ans.* 69° 31′ 40″ N.

## QUADRANTAL TRIANGLES.

---

A triangle is said to be *quadrantal* when one of its sides is $90°$, or a *quadrant*.

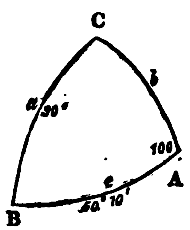
*All* the foregoing rules for right-angled triangles may be applied to quadrantal triangles, with the following exceptions:—

I. The quadrant being left out of the question, the *five circular parts* are the two angles *adjacent* to the quadrant, and the *complements* of the remaining three parts.

II. When two sides or two angles come together on the same side of the equation, the sign − must be prefixed, and the three signs thus placed on one side of the equation must be made to produce the same result, positive or negative, as the sign of the other side.

### Example.

Given $a = 90°$, $A = 100°$, $c = 50° 10'$; to find the other parts of the triangle.




(continued at page 110.

## Examples for Practice.

1. Given $a = 90°$, $b = 76° 41'$, $c = 49° 23' 15''$; find A, B, and C.

*Ans.* $A = 101° 42' 45''$; $B = 72° 20' 15''$; $C = 48° 1'$.

2. Given $a = 90°$, $B = 74° 36' 30''$, $c = 50° 10'$; find the other parts.

*Ans.* $A = 100°$; $b = 78° 14' 30''$; $C = 49° 8' 15''$.

3. Given $a = 90°$, $B = 80° 10'$, $c = 50° 27'$; find the other parts.

*Ans.* $A = 96° 17' 45''$; $C = 50° 2'$; $b = 82° 26'$.

4. Given $a = 90°$, $A = 101° 42' 15''$, $B = 72° 20' 30''$; find $b$, $c$, and C.

*Ans.* $b = 76° 41'$; $c = 49° 23' 45''$; $C = 48° 1' 50''$.

5. Given $a = 90°$, $B = 45°$, $C = 65° 19' 15''$; find the other parts.

*Ans.* $A = 107° 10' 45''$; $b = 47° 44' 30''$; $c = 72°$.

6. Given the latitude of a place $64° 29' 15''$ N., and Sun's declination $15° 12'$ N.; find the setting amplitude.

*Ans.* W. $37° 30'$ N.

7. Where will the Sun rise (*i.e.*, what is his amplitude) in lat. $50° 48'$ N., when the day is 14 hours long?

*Ans.* E. $19° 4' 15''$ N.

(continued at page 111.

## Quadrantal Triangles—continued.

*To find* B.

$$\overset{+}{\text{Cos. } c} = -\underline{\overset{+}{\text{Tan. B}}} \;.\; \overset{-}{\text{Cot. A}}$$

$$\therefore \quad \overset{+}{\text{Tan. B}} = \frac{\text{Cos. } c}{\text{Cot. A}} = \text{Cos. } c \;.\; \text{Tan. A} \qquad (1)$$

*To find b.*

$$\overset{-}{\text{Cos. A}} = -\overset{+}{\text{Cot. } c} \;.\; \underline{\overset{+}{\text{Cot. } b}}$$

$$\overset{+}{\text{Cot. } b} = \frac{\text{Cos. A}}{\text{Cot. } c} = \text{Cos. A} \;.\; \text{Tan. } c. \qquad (2)$$

*To find C.*

$$\overset{+}{\text{Sin. C}} = \overset{+}{\text{Sin. } c} \;.\; \overset{+}{\text{Sin. A}}. \qquad (3)$$

*Calculation.*

| (1) | | (2) | | (3) | |
|---|---|---|---|---|---|
| Tan. B = Cos. *c*. Tan. A. | | Cot. *b* = Cos. A. Tan. *c*. | | Sin. C = Sin. *c*. Sin. A. | |
| Tan. A | 10·753681 | Cos. A . | 9·239670 | Sin. A | 9·993351 |
| Cos. *c* . | 9·806557 | Tan. *c* | 10·078753 | Sin. *c* | 9·885311 |
| Tan. B . | 10·560238 | Cot. *b* | 9·318423 | Sin. C | 9·878662 |
| B = 74° 36′ 30″ | | *b* = 78° 14′ 30″ | | C = 49° 8′ 0″ | |

It is to be borne in mind that if Tan. B, or Cot. *b* had been *negative*, the angles under (1) and (2) must have been subtracted from 180° to obtain B and *b* (see Example, p. 106), and that the ambiguity which arises in determining angle C from the sine may be removed, as in the case of a right-angled triangle, by the rule given in note, page 106.

## Examples for Practice.

8. Given the latitude of the place 50° 48′ N., and Sun's declination 18° 28′ N.; find his rising amplitude, time of setting, and length of the day and night.

*Ans.* Amplitude .. . E. 30° 4′ 30″ N.
Time of setting.. $7^h\ 36^m\ 41^s$.
Length of day .. $15^h\ 13^m\ 22^s$.

9. At what time will the Sun rise in latitude 50° 48′ N., when his azimuth is N. 80° E.?

*Ans.* $5^h\ 28^m\ 52^s$, A.M.

10. Find the *mean time* of sunrise and sunset in latitude 50° 48′ N. when the Sun's declination is 18° 28′ N., the equation of time being $10^m$ add. to apparent time.

*Ans.* Time of rising .... $4^h\ 33^m\ 19^s$, A.M.
" setting .... $7^h\ 46^m\ 41^s$, P.M.

11. In what latitude will the Sun rise at $5^h\ 28^m\ 52^s$, A.M., 10° north of east?

*Ans.* 50° 48′ N.

12. The Sun rose E. 37° 30′ N., at a place in lat. 64° 29′ 15″ N.; what was his declination?

*Ans.* 15° 12′ N.

13. What is the length of the day when the Sun sets W. 19° 5′ N., at a place in lat. 50° 48′ N.?

*Ans.* 14 hours.

## THE SUPPLEMENTAL TRIANGLE

*Three angles of a spherical triangle being given, to find the sides.*

For example, suppose the angles A, B, C are given to find the sides *a, b, c*; let A′ B′ C′ be the supplemental triangle (see figure 9, page 44). Then B′ C′ = 180° − A, A′ C′ = 180° − B, A′B′ = 180° − C; we thus find the three sides of the supplemental triangle, and can therefore calculate the angles A′, B′, C′ by Rule I. Then,

BC = 180° − A′; AC = 180° − B′; AB = 180° − C′.

### Example.

Given A = 59° 2′, B = 74° 54′, C = 71° 18′; to find BC.

| | | | | | |
|---|---|---|---|---|---|
| | 180° 0′ | | 180° 0′ | | 180° 0′ |
| | 59 2 | | 74 54 | | 71 18 |
| B′C′ | 120 58 | A′C′ = | 105 6 | A′B′ = | 108 42 |

*To find* A′ *by* Rule I.

| | | | |
|---|---|---|---|
| 105° 6′ | Cosec. | .... | ·015260 |
| 108 42 | Cosec. | .... | ·023554 |
| 3 36 | | | |
| 120 58 | | | |
| 124 34 | ½ Hav. | .... | 4·947070 |
| 117 22 | ½ Hav. | .... | 4·931614 |
| | | | 9·917498 |

A′ = 130° 50′ 30″
180

BC = 49 9 30

## Examples for Practice.

1. Given $A=130^\circ\ 50'$, $B=121^\circ\ 35'$, $C=123^\circ\ 18'$; find the sides.

*Ans.* $a = 120^\circ\ 58'$, $b = 105^\circ\ 6'$, $c = 108^\circ\ 41'\ 30''$.

2. Given $A = 115^\circ\ 38'\ 45''$, $B = 99^\circ\ 21'\ 15''$, $C = 75^\circ\ 31'\ 30''$; find the sides.

*Ans.* $a = 119^\circ\ 42'\ 20''$, $b = 108^\circ\ 4'\ 18''$, $c = 68^\circ\ 53'\ 42''$.

3. Given $A = 60^\circ\ 17'\ 45''$, $B = 71^\circ\ 55'\ 45''$, $C = 111^\circ\ 6'\ 15''$; find the sides.

*Ans.* $a = 64^\circ\ 21'\ 15''$, $b = 80^\circ\ 38'\ 45''$, $c = 104^\circ\ 28'\ 30''$.

4. Given $A=59^\circ\ 2'$, $B=74^\circ\ 54'$, $C=71^\circ\ 18'\ 30''$; find the sides.

*Ans.* $a = 49^\circ\ 10'$, $b = 58^\circ\ 25'$, $c = 56^\circ\ 42'$.

5. Given $A=81^\circ\ 24'\ 15''$, $B=61^\circ\ 31'\ 30''$, $C=102^\circ\ 59'$; find the sides.

*Ans.* $a=87^\circ\ 10'\ 15''$, $b=62^\circ\ 36'\ 45''$, $c=100^\circ\ 10'\ 15''$.

6. Given $A=101^\circ\ 42'\ 45''$, $B=72^\circ\ 20'\ 15''$, $C=48^\circ\ 1'\ 15''$; find the sides.

*Ans.* $a = 90^\circ$, $b = 76^\circ\ 41'$, $c = 49^\circ\ 23'\ 15''$.

7. Given $A = 100^\circ$, $B = 74^\circ\ 36'\ 30''$, $C = 49^\circ\ 8'\ 15''$; find the other parts.

*Ans.* $a = 90^\circ$, $b = 78^\circ\ 14'\ 30''$, $c = 50^\circ\ 10'$.

8. Given $A=96^\circ\ 17'\ 45''$, $B=80^\circ\ 10'$, $C=50^\circ\ 2'$; find the sides.

*Ans.* $a = 90^\circ$, $b = 82^\circ\ 26'$, $c = 50^\circ\ 27'$.

# MISCELLANEOUS PROBLEMS IN PLANE TRIGONOMETRY.

*For an explanation of the method of solving Problems in Plane Trigonometry, see* APPENDIX.

1. At 120 feet distance from the base of a lighthouse, the angle of elevation of the top was found to be $60^\circ\ 30'$; required the height of the lighthouse.

*Ans.* 212·09 feet.

2. An observer on the same horizontal plane with the foot of a flagstaff 37 ft. 9 in. high finds the elevation of its top to be $11^\circ\ 39'\ 45''$; required his distance from the foot of the flagstaff.

*Ans.* 182·8 feet.

3. A river, A C, the breadth of which is 200 feet, runs at the foot of a tower, C B, which subtends an angle, BAC, of $25^\circ\ 10'$ at the edge of the bank; required the height of the tower.

*Ans.* 93·97 feet.

4. Find the height of a flagstaff which casts a shadow of $24\frac{1}{2}$ feet when the Sun's altitude is $57^\circ\ 3'$.

*Ans.* 37·8 feet.

5. Find the height of an upright object which casts a shadow of 75 ft. 8 in. when the Sun's altitude is $46^\circ\ 58'\ 45''$.

*Ans.* 81·1 feet.

6. Find the distance of a ship at sea when the angle of *depression* of her hull taken from a height of 395 feet is $2^\circ\ 16'\ 15''$.

*Ans.* 9961 feet.

7. Find the distance of a ship at sea when the angle of *depression* taken from a height of 700 feet is $4^\circ\ 0'\ 15''$.

*Ans.* 10000 feet.

8. Find the horizontal distance of an object when its angle of *depression* taken from a height of 208 feet is $5^\circ\ 2'\ 45''$.

*Ans.* 2355 feet.

9. From the top of a cliff, 326 feet above the sea, the angle of depression of a ship's hull was found to be $24^\circ$; required the distance of the ship.

*Ans.* 732·2 feet.

10. Being ordered away to lay out a target at the distance of 1400 yards from my ship, whose masthead is 180 feet above the water-line, at what angle must I set the index of my sextant?

*Ans.* $2^\circ\ 27'\ 15''$.

11. Being ordered away to lay out a target at the distance of 1500 yards from my ship, whose masthead is 187½ feet above the water-line, at what angle must I set the index of my sextant?

*Ans.* $2^\circ\ 23'\ 15''$.

12. Being ordered away to lay out a target at the distance of 1200 yards from my ship, whose masthead is 157 feet above the water-line, at what angle must I set the index of my sextant?

*Ans.* $2^\circ\ 29'\ 45''$.

13. The elevation of a steeple is $13^\circ 48'$, and 165 yards nearer it is $25^\circ 44' 30''$; find the height of the steeple.

*Ans.* 248·16 feet.

14. A person standing on one bank of a river observes the angle of elevation of a tree on the opposite bank to be $55^\circ$; receding 30 feet he finds it to be $48^\circ$; determine the breadth of the river.

*Ans.* 104·9 feet.

15. Sailing away from a mountain peak at the rate of 9 knots an hour, I observed its angle of elevation to be $6^\circ 59' 30''$, 10 minutes afterwards I observed it again to be $2^\circ 1'$; required the height of the mountain.

*Ans.* ·07412 mile.

16. Sailing towards a headland, I observe the elevation of its summit to be $4^\circ 10'$, and 2 knots nearer, $10^\circ$; required its height.

*Ans.* 503·3 yards.

17. Two observers on the same side of a balloon, and in the same vertical plane with it, a mile apart, find the angles of elevation of the balloon to be $15^\circ$ and $65^\circ 30'$, at the same instant; find the height of the balloon.

*Ans.* 537·187 yards.

18. From the top of a cliff I observe two ships in a straight line before me, a mile apart, and find their angles of depression to be $5^\circ$ and $15^\circ$, respectively; find the height of the cliff.

*Ans.* 228·631 yards.

19. A person on the top of a tower, the height of which is 50 feet, observes the angles of depression of two objects to be $30^\circ$ and $45^\circ$, respectively; find their distance from each other.

*Ans.* 36·6 feet.

20. From the top of a hill 520 feet high, I observe two objects in a straight horizontal line directly before me, and observe their angles of depression to be $26^\circ\ 1'\ 30''$ and $19^\circ\ 14'\ 45''$; required the distance between the objects.

*Ans.* 424·4 feet.

21. From the top of a hill 500 feet high, I observe two objects in a straight horizontal line directly before me, and observe their angles of depression to be $27^\circ\ 0'\ 45''$ and $21^\circ\ 48'$; find the distance between the objects.

*Ans.* 269·3 feet.

22. Coming within sight of two headlands bearing North and South of each other, the southern one bore from the ship due East, and the other N.E. by E.; after sailing due East five miles, the northern one bore N.E. $\frac{1}{2}$ N.; required the distance of the headlands from each other, and their distance from the ship at each time of observing them.

*Ans.* 13·31 miles; 9·56 miles; 6·07 miles; and 7·39 miles.

23. Having dropped anchor in a deep river, with a buoy and rope of 50 fathoms attached to it, I fell down the stream till I had veered 140 fathoms of cable, and then the buoy was 120 fathoms ahead of me; required the depth of water.

*Ans.* 48·7 fathoms.

24. The angle of elevation of a tower at the distance of 275 yards W.S.W. from its foot is $7^\circ\ 8'$; what angle does the tower subtend at a spot 380 yards S.S.E. of the former position?

*Ans.* $4^\circ\ 11'\ 45''$.

25. At a distance of 200 yards from the bottom of a church tower, the angle of elevation of the top of the tower was $30^\circ$, and of the top of the spire $32^\circ$; find the height of tower and spire.

*Ans.* 115·47 yards, and 9·503 yards.

26. From the summit of a lighthouse 85 feet high, standing on a rock, the angle of depression of a ship was $3^\circ\ 38'$, and at the bottom of the lighthouse it was $2^\circ\ 43'$; find the horizontal distance of the ship, and the height of the rock.

*Ans.* 5296·4 feet, and 251·3 feet.

27. A flagstaff A, on the top of a hill, is visible from two stations, B and C, 182 yards apart; the altitude of A, taken from B, was $5^\circ\ 35'$, and the angles ABC, ACB, are $77^\circ\ 52'$, and $88^\circ\ 18'$, respectively; find the height of the hill.

*Ans.* 74 yards.

28. In order to ascertain the height of a mountain, a base of 2761 feet was measured, and at either extremity of this base were taken the angles formed by the summit and the other extremity; these were $58^\circ\ 29'$ and $111^\circ\ 52'$; also at the extremity from which the latter was taken, the angular height of the mountain was $11^\circ\ 18'$; required, the mountain's height.

*Ans.* 2751·3 feet.

29. Two ships, half a mile apart, find that the angles subtended by the other ship and a fort are respectively $56° 19'$ and $63° 41'$; find the distance of each ship from the fort.

*Ans.* 910·82 yards; and 845·54 yards.

30. In order to determine the breadth of a river, I measured a base of 500 feet, close to one side of it, and at each extremity of the base found the angle subtended by the other extremity and a tree on the opposite bank to be $53°$, and $79° 12'$, respectively; find the breadth of the river.

*Ans.* 529·5 feet.

31. Wishing to ascertain the height of a tower standing on a declivity, I ascend to within 80 feet of its base, and it then subtends an angle of $30°$; if the inclination of the declivity be $15°$, what is the height of the tower?

*Ans.* 56·57 feet.

32. An object on shore bears from a ship N.W. by N.; after sailing N.E. 5·7 miles, it bears W. by N.; find the distance of the ship at both observations.

*Ans.* 7·906 miles; and 6·702 miles.

33. A point of land bears from a ship N.E. by E., distant 15 miles; she sails E. by S., till it bears N.N.W.; find the distance of the ship from the land, at the second observation.

*Ans.* 12·76 miles.

34. Two ships are observed to bear N.N.E. $\frac{1}{2}$ E., and N.W. by N., distant respectively 3, and 4·8 miles; find the distance between them.

*Ans.* 4·297 miles.

35. From a ship two headlands were observed to bear S.W. and S.E. by E.; after sailing due East 10 miles, they were observed to bear W.S.W. and S.W. by W., respectively; required the distance between the headlands.

*Ans.* 12·63 miles.

36. From a boat I observed that two objects on shore, E. and W. of each other and 500 yards apart, bore E. by N. and N. by W.; required the distance of the boat from each object.

*Ans.* 97·54 yards; 490·4 yards.

37. Two observers from stations 800 yards apart, and lying due North and South, take, at the same instant, the bearings of a balloon; the observer at the northern station finds the bearing to be E.S.E., the other observer finds it to be E.N.E.; the altitude of the balloon at the former station was 60°; what is its height?

*Ans.* 1810 yards.

38. If a balloon, seen from a certain station, has an altitude of 60°, and bearing N.W., what will be its bearing at a station South of the former, when the altitude of the balloon is 45°?

*Ans.* N.N.W $\frac{1}{4}$ W.

39. A mountain seen from a certain station has an altitude 45°, and bearing East; what will be its altitude at a station S.W. of the former, when the bearing is E.N.E.?

*Ans.* 28° 25′ 15″.

40. Two objects, P and Q, were observed from a ship to be at the same instant in a line bearing N. 15° E.: after sailing N.W. 5 miles, P bore E. 11° N., and Q bore N. 41° E.; find the distance between P and Q.

*Ans.* 6·766 miles.

41. A ship at anchor observes two small islands on the same line of bearing, N.N.E. To determine their distance apart, she gets under weigh, and sails due east 5 miles, when one bore N.W., and the other W.N.W.; what was their distance from each other?

*Ans.* 1·91 miles.

42. Two headlands are observed by a ship sailing S.S.W.; one bears from the ship S.W., and the other S.S.E.; after the ship has sailed six miles, the bearing of the first is W. by S., and of the second E.S.E.; find the bearing and distance from each other.

*Ans.* N. 81° 10′ W. 9·71 miles.

43. Two islands, A and B, bear in one line W. 56° S., and after sailing W.N.W. 10·5 nautical miles, they bear S. and S. 21° 30′ W. respectively; required the distance between them.

*Ans.* 31·16 miles.

44. Two ports bear from one another N.E. by E., and S.W. by W., distant 12 miles. A ship has sailed S.S.E. from the more southerly port 15·5 miles; required her bearing and distance from the more northerly port.

*Ans.* S. 10° 54′ 45″ W. 21·38 miles.

45. Two headlands were observed to bear N. 8° W., and N. 21° E.; after sailing E.N.E. 5 nautical miles, the bearings became W. 36° N. and North respectively; find the distance between the headlands, and their bearing one from the other.

*Ans.* 8·226 miles. S. 41° 26′ 30″ W.

46. The bearings of two lighthouses from a ship were W. 49° S., and E. 54° S., after she had sailed North 13 miles the bearings were S. 22° W. and S. 17° E.; find their bearing and distance from each other.

*Ans.* S. 83° 42′ W. 16·78 miles.

47. From a ship two rocks were observed to bear in one line W. 22° S.: after she had sailed S. 5° E. 13·8 miles, they bore N. 39° W. and W. 39° N.; find the distance between them.

*Ans.* 3·281 miles.

48. Two headlands bore from a ship N. 54° E., and S. 81° E.; after sailing S.E. 12 miles, they bore N. 5° W., and E. 40° N.; required their bearing and distance from each other.

*Ans.* N. 47° 7′ W., or S. 47° 7′ E., 11·41 miles.

49. Sailing between two islands I observed the North point of one to bear S. 40° W., and the South point of the other W. 30° N.; after running W. ½ S. 16 miles, the bearings became S. 31° E., and E. 49° N.; what was their bearing and distance from each other?

*Ans.* N. 0° 27′ 15″ E., or S. 0° 27′ 15″ W., 17·32 miles.

50. Being within sight of my port, bearing N.N.E. 18 miles, a fresh gale sprang up at N.E., then running 48 miles on the port tack within 6 points, I went about; required the course and distance to my port.

*Ans.* N. 46° 56′ 30″ W., 51·27 miles.

51. The wind is N.E. ¼ N., and a ship is bound to a port 80 miles directly to windward, which she proposes to reach in two tacks within 6 points of the wind; required the course and distance on each tack so as to double a rocky shelf lying in the S.E. quarter.

*Ans.* Starboard tack, N.N.W. ¼ W., 104·5 miles.
Port ,, E.S.E. ¼ E., 104·5 ,,

52. A privateer lying-to, with the wind at N.N.E., sees a sloop on the starboard tack, that had just doubled a point 18 miles to the W. by N. The sloop ran for her port close hauled (within 6 points). The privateer gave chase at the rate of 8 knots an hour, and in four hours came up with her; required the sloop's rate of sailing and the course of each vessel.

*Ans.* Sloop sails N.W. 3·86 miles per hour. Privateer, N. 63° 12′ W.

53. A headland bears S. 59° E., and a current runs S. by W. 4½ miles an hour; how must I steer to fetch the headland, steaming at the rate of 9½ knots an hour?

*Ans.* S. 85° 28′ E.

54. A lightship bears W. by N. ½ N. A current runs S.W. ¼ W. three miles an hour. How must I steer to fetch the light, steaming at the rate of 10½ knots an hour?

*Ans.* N. 58° 56′ W.

55. A ship in crossing the mouth of a river, out of which the tide sets due east, sails from a buoy on the south side N.E. 10 miles, and then falls in with another buoy on the North side, distant from the first 15 miles; required the ship's course and drift of the current.

*Ans.* Course, N. 61° 32 E. Drift, 6·16 miles.

56. Sailing E. by S. $7\frac{1}{2}$ knots an hour, two islands were observed, one bearing S.E. by S., and the other E. by N. The tide was then running S.W. by W. $2\frac{1}{2}$ miles an hour: two hours afterwards the first island bore W. by S., and the other N.W. by N.; required the ship's course and distance, together with the bearing and distance of these islands from one another.

*Ans.* Ship's course S. $61^\circ 37'$ E. Distance 12 miles. The N. isle bore from S., N. $9^\circ 30'$ E., 8·116 miles.

57. If the height of the Peak of Teneriffe be 4 miles, and the angle taken at the top of it, as formed between a plumb line and a line conceived to touch the horizon, or farthest visible point, be $87^\circ 25' 55''$; it is required to determine the diameter of the Earth, and the utmost distance that can be seen on its surface from the top of the mountain, supposing the form of the Earth to be perfectly spherical.

*Ans.* Diam. 7957·818 miles. Dist. 178·458 miles.

58. From the top of a mountain $1\frac{1}{2}$ miles high, the dip of the horizon was found to be $1^\circ 34' 30''$; what is the diameter of the Earth?

*Ans.* 7937·86 miles.

59. A fleet steering S.W. by S. 4 miles an hour, on discovering a sail, detached a frigate which gives chase S.E. 5 hours, at the rate of 7 knots an hour, and comes up with and takes the chase. After an hour's time spent in adjusting matters on board the prize, she steers for the fleet, which still kept on the same course and at the same rate; required the course the frigate must shape, and the distance she must run at 7 miles an hour, to rejoin the fleet.

*Ans.* She must steer S. $66^\circ 26'$ W., and run 63·58 miles.

# MISCELLANEOUS PROBLEMS IN NAUTICAL ASTRONOMY, &c.

*For an explanation of the method of solving these Problems, see* APPENDIX.

1. Prove that the altitude of the pole is equal to latitude of the place.

2. What is the latitude of a place at which the Sun's meridian altitude is observed to be 30° when its declination is 23° S.?

*Ans.* 37° N.

3. What is the altitude of the Sun on the longest day when on the meridian of Portsmouth?

*Ans.* 62° 42′.

4. The declination of a star is 30° N., and it is found to graze the North point of the horizon without setting; what is the latitude of the place?

*Ans.* 60° N.

5. A ship rounding Cape Horn has the Sun 5° above the horizon at noon on the 21st of June; what is her latitude?

*Ans.* 61° 30′ S.

6. A ship observes the Sun at noon just to graze the South point of the horizon without rising when its declination is 20° S.; what is her latitude?

*Ans.* 70° N.

7. The star $\alpha$ Ursæ Majoris has an altitude of $10^\circ$ when on the meridian below the pole, and 12 hours after it has an altitude of $80^\circ$; what is the latitude?

*Ans.* $45^\circ$ N.

8. A star has an altitude of $5^\circ$ when on the meridian below the pole, and 12 hours after its altitude is $75^\circ$; what is the latitude?

*Ans.* $40^\circ$ N.

9. In latitude $40^\circ\ 25'\ 10''$ N., when the Sun's declination was $18^\circ\ 2'\ 53''$ N., his true altitude was $35^\circ\ 26'\ 34''$ (West of Meridian); find the apparent time.

*Ans.* $3^h\ 53^m\ 43^s$.

10. In latitude $15^\circ\ 23'$ N., the Sun's declination was $8^\circ\ 20'\ 45''$ S., and his altitude $15^\circ\ 25'\ 36''$ (West of Meridian); find the apparent time.

*Ans.* $4^h\ 45^m\ 33^s$.

11. Given latitude $48^\circ\ 20'$ N., Sun's true altitude $15^\circ\ 51'$ (West of Meridian), and declination $6^\circ\ 37'\ 15''$ S.; find his azimuth.

*Ans.* N. $119^\circ\ 57'\ 30''$ W.

12. Given latitude $30^\circ\ 17'$ S., Sun's true altitude $18^\circ\ 44'\ 45''$ (East of Meridian), and declination $10^\circ\ 17'\ 49''$ S.; find the azimuth.

*Ans.* S. $88^\circ\ 50'$ E.

13. Calculate the zenith distance of $\alpha$ Leonis (Regulus) having given latitude of place $45^\circ\ 28'$ N., star's declination $12^\circ\ 46'\ 15''$ N., and hour angle $1^h\ 8^m\ 11^s$.

*Ans.* $35^\circ\ 44'$.

14. Calculate the altitude of the Moon's centre, having given latitude of place, $45^\circ$ $28'$ N., declination $25^\circ$ $8'$ $37''$ N., and hour angle $1^h$ $49^m$ $36^s$.

*Ans.* $60^\circ$ $3'$.

15. If the Sun's declination be $10^\circ$ N., at what time will he appear due East, to an observer in latitude $50^\circ$ N.?

*Ans.* $6^h$ $34^m$ $2^s$ A.M.

16. What is the Sun's declination if his altitude at six o'clock, at a place in latitude $50^\circ$ $48'$ N., is $16^\circ$ $20'$?

*Ans.* $21^\circ$ $16'$ $30''$ N.

17. A star whose declination is $51^\circ$ $30'$ N., was observed when due East, to have an altitude of $64^\circ$ $31'$; find the latitude of the place of observation.

*Ans.* $60^\circ$ $6'$ $15''$ N.

18. The Sun's declination is $14^\circ$ N.; find his azimuth when rising, at a place in latitude $54^\circ$ N.

*Ans.* N. $65^\circ$ $41'$ $45''$ E.

19. What is the Sun's altitude, when his azimuth is $110^\circ$ $4'$ at the time of the equinox, in latitude $54^\circ$ N.?

*Ans.* $13^\circ$ $59'$ $45''$.

20. A star situated on the celestial equator is observed to have an altitude of $60^\circ$, and two hours afterwards to have the same altitude; find the latitude of the place.

*Ans.* $26^\circ$ $17'$ $30''$ N.

21. What is a star's declination when it rises N.E. at a place in latitude $50^\circ$ N.?

*Ans.* $27^\circ$ $2'$ N.

22. The Sun sets S.W., and his declination is 20° S.; find the latitude.

*Ans.* 61° 4′ 30″ N.

23. Where will the Sun set, in latitude 61° 4′ 30″ N., when his declination is 20° S.?

*Ans.* S.W.

24. A star whose declination is 27° 2′ N., is observed to rise N.E. at a certain place; what is the latitude of that place?

*Ans.* 50° N.

25. What is the Sun's altitude if his declination at six o'clock, at a place in latitude 50° 48′ N., is 21° 16′ 30″ N.?

*Ans.* 16° 20′.

26. A star whose declination is 50° 30′ N., was observed when due East, to have an altitude of 64° 51′; find the latitude of the place of observation.

*Ans.* 58° 28′ 45″ N.

27. The Sun's declination is 14° N., and his azimuth when rising is N. 65° 41′ 45″ E.; find the latitude.

*Ans.* 54° N.

28. What is the Sun's azimuth, when his altitude is 13° 59′ 45″ at the time of the equinox, in latitude 54° N.?

*Ans.* 110° 4′.

29. A star situated on the celestial equator is observed to have an altitude of 40°, and four hours afterwards to have the same altitude; find the latitude of the place.

*Ans.* 42° 4′ 45″ N.

30. What is the star's declination when it rises S.E. at a place in latitude 50° S.?

*Ans.* 27° 2′ S.

31. The Sun sets N.W., and his declination is 20° N.; find the latitude.

*Ans.* 61° 4′ 30″ N. or S.

32. Where will the Sun set, in latitude 61° 4′ 30″ S., when his declination is 20° N.?

*Ans.* N.W.

33. A star whose declination is 27° 2′ S., is observed to rise S.E. at a certain place; what is the latitude of that place?

*Ans.* 50° N. or S.

34. How long will the Sun be above the horizon of a place in latitude 50° N. when its declination is 20° N.?

*Ans.* $15^h\ 25^m\ 40^s$.

35. How long will the moon be above horizon of a place in latitude 52° 30′ N., when its declination is 20° S.

*Ans.* $8^h\ 13^m\ 28^s$.

36. Find the time of sunrise at Madeira on the longest day.

*Ans.* $4^h\ 55^m\ 23^s$ A.M.

37. Find the time of sunrise at Naples on the shortest day.

*Ans.* $7^h\ 28^m\ 17^s$ A.M.

38. At what time of the day will the Sun bear S.E. at the time of the equinox at a place in latitude 40° N.?

*Ans.* $9^h\ 49^m\ 4^s$ A.M.

39. At what time A.M. will the Sun be on the twilight circle, 18° below the horizon, in latitude 50° N., when its declination is 0°.?

*Ans.* $4^h\ 5^m\ 4^s$.

40. At the time of the equinox the Sun is observed to have an altitude of 60°, and 3 hours afterwards to have the same altitude; what is the latitude of the place.

*Ans.* 20° 23′ N., or 20° 23′ S.

41. Having given the R. A. of a star $3^h\ 30^m$ and its declination 50° 30′ N., calculate its latitude and longitude.

*Ans.* Lat. 30° 25′ 30″ N. Long. 63° 18′ 45″.

42. Having given the latitude of a star 20° N., and its longitude 75° 40′; calculate its R. A. and declination.

*Ans.* R. A. $4^h\ 46^m\ 20^s$. Dec. 42° 35′ N.

43. Calculate the distance between *a* Arietis and the Moon, having given star's R. A. $1^h\ 59^m\ 22^s$, star's declination 22° 49′ 4″ N., Moon's R. A. $9^h\ 11^m\ 7^s$, Moon's declination 11° 8′ 36″ N.

*Ans.* 101° 44′ 44″.

44. Calculate the distance between the Sun and Moon, having given Sun's R.A. $14^h\ 40^m\ 36^s$, declination 15° 37′ 50″ S., Moon's R. A., $10^h\ 11^m\ 53^s$, declination 5° 32′ 27″ N.

*Ans.* 69° 46′ 23″.

45. Compute the latitude from the following data:—

| Aldebaran. | | Pollux. | |
|---|---|---|---|
| R.A. ...... | $4^h\ 28^m\ 14^s$ | R.A. ...... | $7^h\ 37^m\ 7^s$. |
| Dec. .... | 16° 14′ 12″ N. | Dec. ...... | 28° 20′ 49″ N. |
| True Alt. | 66° 10′ | True Alt. .. | 58° 49′ |

*Ans.* 37° 49′ N.

46. Compute the true distance between the Sun and Moon, from the following elements:—

| | For the Sun. | For the Moon. | App. dist. of their centres. |
|---|---|---|---|
| App.Z.D... | 59° 30′ 12″ .. | 39° 5′ 22″ .. | 88° 49′ 58″ |
| TrueZ.D... | 59° 31′ 44″ .. | 38° 30′ 40″ | |

*Ans.* 88° 24′ 27″.

47. Find the distance on the arc of a great circle and the latitude and longitude of the vertex—

(1) Between Bermuda (St. George's) and the Lizard.

*Ans.* Distance 2796 miles. Lat. of Vert. 49° 58′ N.
Long. of Vert. 6° 54′ W.

(2) Between Cape Clear and St. John's, N. F.

*Ans.* Distance 1674 miles. Lat. of Vert. 52° 1′ N.
Long. of Vert. 21° 22′ W.

(3) Between Cape of Good Hope and Cape Horn.

*Ans.* Distance 3586 miles. Lat. of Vert. 57° 49′ S.
Long. of Vert. 46° 1′ W.

(4) Between Cape of Good Hope and Monte Video.

*Ans.* Distance 3586 miles. Lat. of Vert. 41° 1′ S.
Long. of Vert. 19° 28′ W.

(5) Between Cape of Good Hope and Rio Janeiro.

*Ans.* Distance 3266 miles. Lat. of Vert. 34° 48′ S
Long. of Vert. 9° 29′ E.

# PROBLEMS IN ANALYTICAL PLANE TRIGONOMETRY.

1. A string, whose length is $a$, being fastened to the top of a tree, makes an angle $\theta$ with the ground; show that the height of the tree $= a$ Sin. $\theta$.

2. A tower, whose height is $p$, subtends an angle, $\beta$, to an observer stationed at a certain distance from its base; show that that distance $= p$ Cot. $\beta$.

3. If the hypothenuse, base, and perpendicular of a right-angled triangle be represented by $a$, $b$, and $c$ respectively; show that the perpendicular from the right angle on the hypothenuse $= \frac{bc}{a}$.

4. The side of an equilateral triangle is $a$; show that the perpendicular from either of the angles to the opposite side $= \frac{a}{2}\sqrt{3}$.

5. In the right-angled triangle ABC, C being the right angle, show that Sin. 2A, or Sin. 2B $= \frac{2ab}{c^2}$.

6. In the same triangle, show that Sin. (A − B) $= \frac{a^2-b^2}{c^2}$; and Cos. (A − B) $= \frac{2ab}{c^2}$; or, $= \frac{2ab}{a^2+b^2}$.

7. Standing at a distance of $n$ yards from the foot of a church tower, I observe the angle of elevation of the top of the spire to be $a$, and that of the top of the tower to

be $\beta$; show that the height of the spire $= n\,(\text{Tan. } a - \text{Tan. } \beta)$;
or, $= \dfrac{n\,.\,\text{Sin. } (a-\beta)}{\text{Cos. } a\,.\,\text{Cos } \beta}$.

8. Standing in a line between two monuments on a horizontal plane, I find the angle of elevation of the taller one $a$, to be $\theta$, and of the shorter one $b$, to be $\phi$; show that their distance apart $= a \text{ Cot. } \theta + b \text{ Cot. } \phi$.

9. If, in the above question, $\epsilon$ be the angle of elevation of the top of the taller one above that of the shorter, show that $\text{Tan. } \epsilon = \dfrac{a-b}{a \text{ Cot. } \theta + b \text{ Cot. } \phi}$.

10. If $a$ and $\beta$ be the angles at the base $c$ of any acute-angled triangle, $a$ and $b$ the sides respectively opposite to them; show that $c = b\,.\,\text{Cos. } a + a\,.\,\text{Cos. } \beta$.

11. If $p$ be the perpendicular let fall from the vertical angle to the base of the triangle in the preceding question, show that $p = \dfrac{c}{\text{Cot. } a + \text{Cot. } \beta}$; or, $= \dfrac{c\,.\,\text{Sin. } a\,.\,\text{Sin. } \beta}{\text{Sin. } (a+\beta)}$.

12. A lighthouse has an altitude $a$, and bearing East, to an observer stationed at a certain distance from the base, and at a second station, $a$ miles South of the former, the angle between the base of the lighthouse and the first station was also $a$: show that the height of the lighthouse $= a \text{ Tan.}^2 a$; and that its distance from the second station $= a \text{ Sec. } a$.

13. A balloon seen from a certain station has an altitude of 50°, and bearing N.W.; what will be its bearing at a station South of the former, where the altitude of the balloon is 30°?

*Ans.* N. by W. $\frac{3}{4}$ W.

14. A mountain seen from a certain station has an altitude of $15^\circ$, and bearing E. by N.; what will be its altitude at a station S.W. of the former, where the bearing is E.N.E.?

*Ans.* $10^\circ 27' 30''$.

15. The top of a lighthouse has an elevation of $30^\circ$, and bearing East to an observer at A; after sailing due South to B, its elevation was found to be $18^\circ 35'$; show that its bearing at B was N.E. by N. nearly.

16. To determine the breadth of a river, I observed the angle of elevation of a tower on the opposite bank to be $a$, and on receding $a$ yards, it was found to be $\beta$; show that the river's breadth $= \dfrac{a \,.\, \text{Cos.}\, a \,.\, \text{Sin.}\, \beta}{\text{Sin.}\,(a-\beta)}$.

17. Two observers, $a$ miles apart, and on different sides of it, observe the angles of elevation of a balloon, in the same vertical plane with them, to be $\theta$ and $\phi$ respectively; show that the altitude of the balloon $= \dfrac{a \,.\, \text{Sin.}\, \theta \,.\, \text{Sin.}\, \phi}{\text{Sin.}\,(\theta+\phi)}$.

18. From the top of a tower, whose height is $h$, the angles of depression of two objects, lying in the same horizontal plane with the base of the tower, and in the same direction, are $a$ and $\beta$; show that their distance apart $= h \,.\, \text{Cosec.}\, a \,.\, \text{Cosec.}\, \beta \,.\, \text{Sin.}\,(a \backsim \beta)$.

19. The angle of elevation of the summit of a mountain, taken at its base, was found to be $a$; after walking $b$ yards up its side, which was inclined at an angle, $\beta$, to the horizon, the angle between the summit and the first station was found to be $\gamma$; show that the height of the mountain $= \dfrac{b \,.\, \text{Sin.}\, a \,.\, \text{Sin.}\, \gamma}{\text{Sin.}\,(a-\beta+\gamma)}$.

20. From the top of a tower a person observes the angles of depression of two distant points $a$ yards apart in the horizontal plane to be $\alpha$ and $\beta$, and the angle between them to be $\gamma$; show that the height of the tower

$$= \frac{a}{\sqrt{\text{Cosec.}^2 \alpha + \text{Cosec.}^2 \beta - 2\,\text{Cosec.}\,\alpha\,.\,\text{Cosec.}\,\beta\,.\,\text{Cos.}\,\gamma}}.$$

21. From the top of a mountain, $a$ miles above the sea, the angle of depression of the sea horizon was found to be $\theta$; required the diameter of the earth.

*Ans.* $a\,.\,\text{Cos.}\,\theta\,.\,\text{Cosec.}^2 \dfrac{\theta}{2}$.

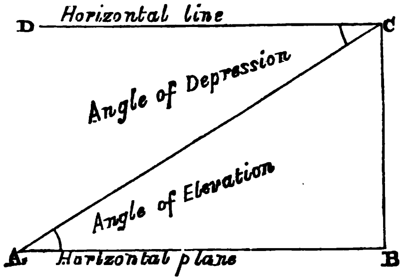
22. Three objects, A, B, C, form an isosceles triangle whose vertex is $\beta$, and whose angles are as the numbers 4, 1, 1; an observer walking from A towards C, measures a base, AD, of $a$ feet, and observes the angle BDC; he then advances to E, $b$ feet further, and observes the angle BEC = the supplement of BDC. From these observations find the sides of the triangle.

*Ans.* $\dfrac{2\sqrt{3}}{3}\left(a + \dfrac{b}{2}\right)$; or $(2a + b)\,.\,\dfrac{\sqrt{3}}{3}$.

23. A person walking from C to D on the horizontal road, can plainly see the summit of a hill, A, from every point except E, where he can just see over it a hill, B. He measures EC, and at C observes the angles of elevation of B and A, as well as the angles ACB, ACE. At E he observes the angle AEC. Show how to find the height of each hill.

# APPENDIX.

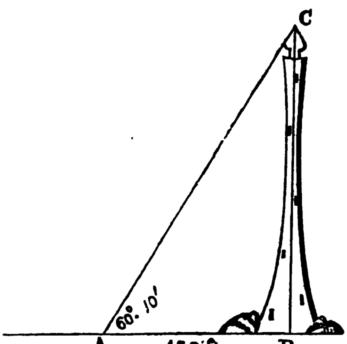
## ANGLES OF ELEVATION AND DEPRESSION.




First, let A be the place of the observer on the horizontal plane AB, C the summit of the object, then CAB is the angle of *elevation* of C.

Secondly, let C be the place of the observer and A the object, CD a horizontal line through the place of the observer, then DCA will be the angle of *depression* of A, and these two angles will be equal, because AB and CD are parallel and AC meets them, therefore they are alternate angles. (Euc. I, 29.)

### On the Solution of Problems in Plane Trigonometry.




**Ex.** 1. At 120 feet distance from the base of a lighthouse, the angle of elevation of the top was found to be 60° 10′; required the height of the lighthouse.

Let CB be the lighthouse, AB, the distance from the base, 120 feet, CAB the angle of elevation equal to 60° 10′.

Then in the right-angled triangle CAB, we have given the base AB and the angle CAB to find the perpendicular CB, which is the height of the lighthouse.

By Rule II, for right-angled triangles,

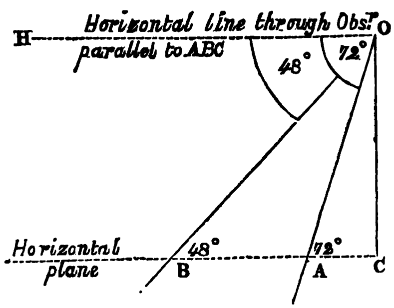
$$\frac{CB}{AB} = \text{Tan. } BAC$$

$$\therefore \quad CB = AB \,.\, \text{Tan. } BAC = 120 \,.\, \text{Tan. } 60^\circ\ 10'.$$

| | |
|---|---|
| Log. 129 .. .. | 2·079181 |
| Log. Tan. 60° 10′ | 10·241483 |
| Log. CB .. .. | 2·320664 |

$$\therefore \quad CB = 209{\cdot}2 \text{ feet.}$$

**Ex.** 2. From the top of a tower whose height is 124 feet, the angles of depression of two objects lying in the same horizontal plane with the base of the tower, and in the same direction, are 72° and 48°, what is their distance apart?




Let OC be the tower, BAC the horizontal plane, O the place of the observer, HO a horizontal line. Make HOB = 48°, and HOA = 72°, then A and B will be the places of the objects.

Now in the figure OBAC, OBA = alt. angle HOB = 48°, and OAC = alt. angle HOA = 72°; therefore we can find AOB; for OAC being equal to ABO + AOB (Euc. I, 32). AOB is equal to OAC − ABO = 72° − 48° = 24°; or BOA = HOA − HOB = 24°.

Hence the three angles being known, if one of the sides OA, or OB were known, we could find AB, the side required.

OA can be found from the right-angled triangle OAC.

$$\text{For } \frac{OA}{OC} = \text{Cosec. } OAC. \quad \therefore \; OA = OC \,.\, \text{Cosec. } OAC.$$

$$\begin{aligned} \text{Log. } OC &= 2{\cdot}093422 \\ \text{Log. Cosec. } OAC &= 10{\cdot}021794 \\ \hline \text{Log. } AO &= 2{\cdot}115216 \end{aligned}$$

$$\therefore \; AO = 130{\cdot}4 \text{ feet.}$$

This determines AO; then we have by the Rule of Sines,

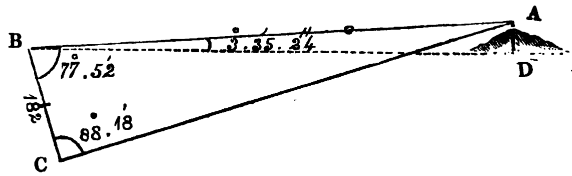
$$AB : AO :: \text{Sin. } AOB : \text{Sin. } OBA.$$

$$\begin{aligned} \text{Log. } AO &= 2{\cdot}115216 \\ \text{Log. Sin. } AOB &= 9{\cdot}609313 \\ \hline &\phantom{=}\; 11{\cdot}724529 \\ \text{Log. Sin. } OBA &= 9{\cdot}871073 \\ \hline \text{Log. } AB &= 1{\cdot}853456 \end{aligned}$$

$$\therefore \; AB = 71{\cdot}36 \text{ feet.}$$

**Ex.** 3. A flagstaff A on the top of a hill is visible from two stations B and C, 182 yards apart; the altitude of A taken from B is $3^\circ\ 35'\ 24''$, and the angles ABC, ACB are found to be $77^\circ\ 52'$ and $88^\circ\ 18'$ respectively; find the height of the hill.

Let AD represent the height of the flagstaff above the horizontal plane BD, and the angle ABD its elevation or altitude: B, C the two stations. At B the angle ABC between the flagstaff and the station C is observed. Also




at C the angle BCA between the flagstaff and the station B is observed; required the height of AD.

1st. In the triangle ABC the side BC and the angles ABC and BCA are given, and BAC being equal to $180^\circ - (77^\circ\ 52' + 88^\circ\ 18')$ or $13^\circ\ 50'$, we can find AB by the Rule of Sines.

$$\text{For AB : BC :: Sin. ACB : Sin. BAC.}$$

$$\textit{i.e.}\ \text{AB} : 182 :: \text{Sin. } 88^\circ\ 18' : \text{Sin. } 13^\circ\ 50'.$$

$$\text{Whence AB} = 760{\cdot}9 \text{ yards.}$$

2nd. In the right-angled triangle BAD, knowing AB and angle ABD, we can find AD.

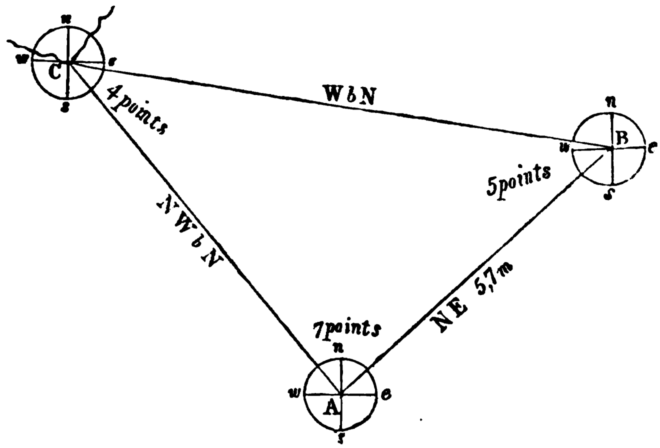
$$\text{For AD} = \text{AB . Sin. ABD.}$$

$$= 760{\cdot}9 \text{ Sin. } 3^\circ\ 35'\ 24''.$$

$$= 47{\cdot}6 \text{ yards.}$$

**Ex. 4. From a ship a cape was observed to bear N.W. by N.: after she had sailed N.E. 5·7 miles, it bore W. by N.; find its distance at both observations.**

**Note.—In those cases in which compass bearings are concerned, it will facilitate the working of the problem if a small compass be drawn at each angle of the triangle, thus—**




Let A be the first position of the ship; describe a small compass, and from its centre draw AC in the direction of N.W. by N., the first bearing, and AB equal to 5·7 miles N.E.; at B describe another compass, and draw BC in the direction of W. by N.; then where AC and BC cut each other will the position of the Cape.

To find AC and BC, the distance of the ship at each observation, we have given BAC, the angle between N.E. and N.W. by N., equal to 7 points, and ABC the angle between N.E. (or its opposite bearing S.W.) and W. by N., equal to 5 points, and AB equal to 5·7 miles.

BCA = 180° − (BAC + ABC) = 180° − (7 pts. + 5 pts.) = 4 pts.

∴ By the Rule of Sines,

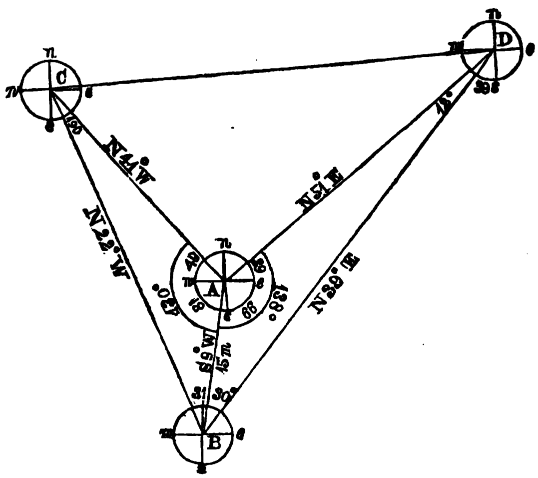
AC:AB::Sin.ABC:Sin.BCA. And BC:AB::Sin.BAC:Sin.BCA.

| | |
|---|---|
| 0·755875 | 0·755875 |
| 9·919846 | 9·991574 |
| 9·675721 | 10·747449 |
| 9·849485 | 9·849485 |
| 0·826236 | 0·897964 |

∴ AC = 6·7 miles, and BC = 7·9 miles.

**Ex. 5.** Two headlands bore from a ship N. 41° W., N. 51° E. After she had sailed S. 9° W. 15 miles, they bore N. 22° W. and N 39° E. Required their bearing and distance from each other.

Describe a small compass at A, the first position of the ship, and measuring 41° from *n* towards *w*, lay off the line



of bearing AC, and at 51° from *n* towards *e*, the second line of bearing AD. Also lay off AB, 9° from *s* towards *w*, and let it equal 15 miles, the run of the ship. At B describe another compass, and lay off the two lines of bearing BC N. 22° W. and BD, N. 39° E., then C and D will be the positions of the two headlands.

It is required to find *s*DC or *n*CD, their bearing from each other, and CD their distance apart.

1st. In triangle BAD, we have given angle BAD $= 9° + 90° + 39° = 138°$; ABD $= 39° - 9° = 30°$; ADB $= 180° - (138° + 30°) = 12°$, and AB = 15m, to find BD by the Rule of Sines.

$\therefore$ BD : AB :: Sin. BAD : Sin. ADB

*i.e.* BD : 15 :: Sin. 138° : Sin. 12°

Whence BD = 48·28 miles.

2nd. In triangle ABC, we have given CAB $= 81° + 49° = 130°$; CBA $= 22° + 9° = 31°$; ACB $= 180° - (130° + 31°) = 19°$; and BC = 15m; to find BC as before.

$\therefore$ BC : AB :: Sin. CAB : Sin. ACB

$\therefore$ BC : 15 :: Sin. 130° : Sin. 19°

Whence BC = 35·3 miles.

3rd. Then in triangle DBC, we have given the two sides, DB, BC, and the included angle DBC (equal to $30° + 31°$, or $61°$), to find BDC by the Rule of Tangents; this added to SDB will give SDC, the bearing of C from D.

$$\therefore \text{Tan.} \tfrac{1}{2}(C-D) : \text{Tan.} \tfrac{1}{2}(C+D) :: BD-BC : BD+BC.$$

$$\begin{array}{lrr} CBD = & 61° & 0' \\ & 180 & \\ \hline 2 \mid & 119 & 0 \\ \hline \tfrac{1}{2}(C+D) = & 59 & 30 \end{array} \qquad \begin{array}{rl} BD = & 48{\cdot}28 \\ BC = & 35{\cdot}30 \\ \hline BD - BC = & 12{\cdot}98 \\ BD + BC = & 83{\cdot}58 \\ \hline \end{array}$$

$$\therefore \text{Tan.} \tfrac{1}{2}(C-D) : \text{Tan.}\ 59° 30' :: 12{\cdot}98 : 83{\cdot}58.$$

$$\begin{array}{r} 10{\cdot}229852 \\ 1{\cdot}113275 \\ \hline 11{\cdot}343127 \\ 1{\cdot}922102 \\ \hline 9{\cdot}421025 \\ \hline \end{array}$$

$$\begin{array}{lrrr} \therefore \tfrac{1}{2}(C-D) = & 14° & 46' & 15'' \\ \tfrac{1}{2}(C+D) = & 59 & 30 & 0 \\ \hline \therefore D, \text{ or } CDB = & 44° & 43' & 45'' \\ \hline \end{array}$$

*To find the Bearing.*

$$\begin{array}{lrrr} CDB = & 44° & 43' & 45'' \\ SDB = & 39 & 0 & 0 \\ \hline \therefore SDC = & 83 & 43 & 45 \\ \hline \end{array}$$

Therefore C bears from D S. 83° 43′ 45″ W.

4th. Lastly, in triangle CBD we have given two angles, CDB, CBD, and the side CB, to find CD by the Rule of Sines.

$$\therefore CD : CB :: \text{Sin. } CBD : \text{Sin. } CDB.$$

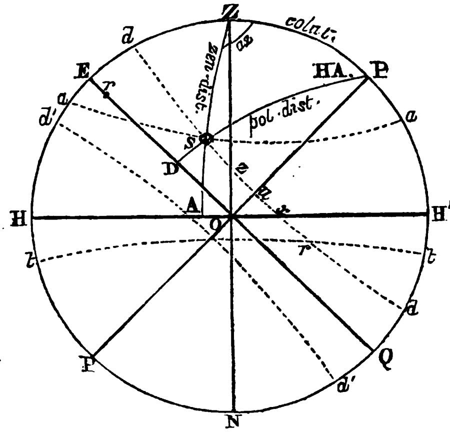
$$i.e.\ CD : 35{\cdot}3 :: \text{Sin. } 61° : \text{Sin. } 44° 43' 45''.$$

Whence CD = 43·87 miles, their distance apart.

*Ans.* Bearing S. 83° 43′ 45″ W. Distance 43·87 miles.

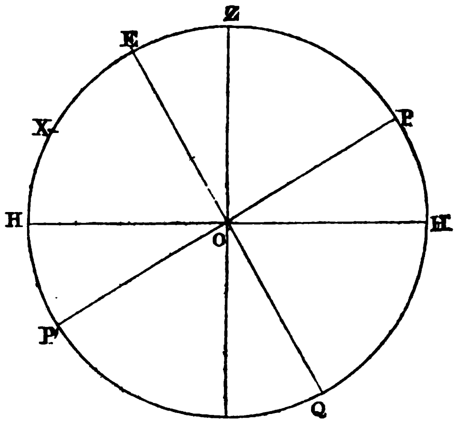
## EXPLANATION OF ASTRONOMICAL TERMS.

No difficulty will be found in working the astronomical problems given in this book, if attention be paid to the explanation of the accompanying figure:—




O in the centre of the figure is the place of the spectator: HH′, his horizon: HZH′N, the celestial meridian: EQ the equator: P, P, the poles: PP, the six o'clock hour circle: Z, the zenith: N, the nadir: ZN, the prime vertical: *a a*, a parallel of altitude: *d d*, a parallel of North declination: *d′d′*, a parallel of South declination. These also represent the diurnal paths of all bodies which may be situated upon them; thus, if E *d* were the declination of a body, *d d* is its daily path; *r*, its place on the twilight circle, *t t*; *x*, its place at rising; *o x* its rising amplitude; *u*, its place on the six o'clock hour circle; *z*, its place on the prime vertical, when it bears due East or West; *d*, its place on the meridian at noon; *d* (below the horizon) its place at midnight; *x z d* half the length of the day; *x r d*, half the length of the night; P S D, a circle of declination; Z S A, a circle of altitude; S D, the declination of S; S A, its altitude; P S, polar distance; Z S, zenith distance; Z P S, hour angle; P Z S, (or H′ A) azimuth; Z P, colatitude; Z E or P H, latitude; ♈ D, right ascension. (It is to be borne in mind that Z N, E Q, P P, H O, are circles and not straight lines, but owing to the manner in which the figure is projected we only see their planes.) H′, is the north point of the horizon; H, the south; O, the east; and the point opposite to it, the west. When south latitude is represented the figure should be drawn with P P inclined to the left, or P over the *south* point of the horizon.

**Ex.** 1. **To prove that the altitude of the pole is equal to the latitude of the place.**




In the fig. ZE is the arc of the meridian intercepted between the zenith of the place and the equator, and is therefore equal to the latitude, also PH is the altitude of the pole.

| | |
|---|---|
| Now, | $PE = 90°$ |
| and | $ZH = 90°$ |
| $\therefore$ | $PE = ZH,$ |

Take away the common part ZP.

| | |
|---|---|
| Then, | $ZE = PH$ |
| or, | Lat. = alt. of pole. |

**Ex.** 2. Given the latitude of a place, 30° N., and the altitude of a heavenly body 35° when on the meridian; find the declination.

In the same figure, let ZE = 30°, and XH, the altitude of the body = 35°, then, to find EX, we have

$$\begin{aligned} EX &= ZH - (ZE + XH) \\ &= 90° - (30° + 35°) \\ &= 25° \end{aligned}$$

And as X is south of the equator, the declination is 25° S.

**Ex.** 3. Given lat. 45° 30′ N., Sun's altitude 40° 20′, and declination 15° 30′ N.; to find the apparent time.

(See Figure, p. 144.)

Draw the meridian ZNH′, and the horizon HH′; take H′P equal to the latitude, 45° 30′, and draw PP; make PE equal to 90°, and draw the equator EQ; let H$a$ = 40° 20′, the altitude of the body, and E$d$ = 15° 10′, its declination; and draw the parallels *aa, dd*; then where they intersect will be the place of the Sun, as S. Through this place draw the circle of declination PSD, and the circle of altitude ZSA. Then, in the spherical triangle ZPS, we shall have given, ZP the colat., PS the polar dist., and ZS the zen. dist.; to find ZPS, the apparent time by Rule I, p. 90.

*To find the Three Sides.*

| | | | | | |
|---|---|---|---|---|---|
| | 90° 0′ | | 90° 0′ | | 90° 0′ |
| Lat. = | 45 30 | Dec. | 15 30 | Alt. | 40 20 |
| ZP = | 44 30 | PS = | 74 30 | ZS = | 49 40 |

*To calculate the Apparent Time.*

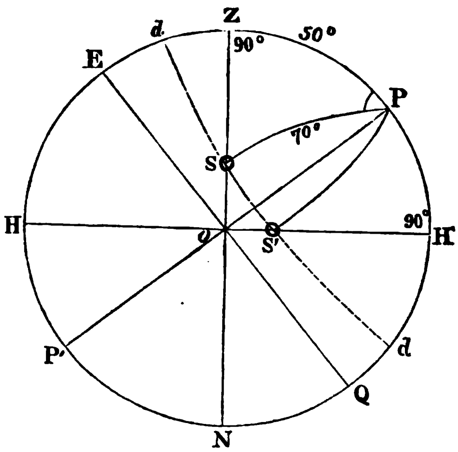
| | | | |
|---|---|---|---|
| ZP .. | 44° 30′ | Cosec. | ·154338 |
| PS .. | 74 30 | Cosec. | ·016090 |
| | 30 0 | | |
| ZS .. | 49 40 | | |
| | 79 40 | ½Hav. | 4·806557 |
| | 19 40 | ½Hav. | 4·232444 |
| | | | 9·209429 |

Apparent time, $3^h\ 9^m\ 51^s$.

If the body is West of the Meridian, this is the time P.M.; if East, it must be subtracted from $24^h$, because it wants so much to noon. This may be accomplished by taking out the time at the *bottom* of the table of Haversines.

It is evident that the Sun's azimuth PZS can be found from the same data as above.

**Ex.** 4. Given lat. 40° N., and declination 20° N.; find the time when the Sun will be on the prime vertical, and its altitude, at that time.




Here ZP = 50°, PS = 70°, and PZS = 90°, because the prime vertical cuts the meridian at right angles.

$$\therefore \underline{\overset{+}{\text{Cos.}}\,P} = \overset{+}{\text{Tan.}}\,ZP\,.\,\overset{+}{\text{Cot.}}\,PS. \quad \text{And } \overset{+}{\text{Cos.}}\,PS = \overset{+}{\text{Cos.}}\,ZP\,.\,\underline{\overset{+}{\text{Cos.}}\,ZS}$$

$$\text{or, Cos. } ZS = \text{Cos. } PS\,.\,\text{Sec. } ZP.$$

| | |
|---|---|
| Tan. ZP | 10·076187 |
| Cot. PS | 9·561066 |
| Cos. P | 9·637253 |
| | $4^h\ 17^m\ 10^s$ P.M. |
| | 12 |
| | 7 42 50 A.M. |

| | |
|---|---|
| Sec. ZP | 10·191933 |
| Cos. PS | 9·534052 |
| Cos. ZS | 9·725985 |
| ZS = | 57° 51′ 15″ |
| | 90 |
| Alt. = | 32 8 45 |

*To find the Amplitude.*

If S′ be the place of the Sun at rising, O S′ is its amplitude, and P H′S′ being a right angle, we have by joining PS′

$$\overset{+}{\text{Cos.}}\,PS' = \overset{+}{\text{Cos.}}\,PH'\,.\,\underline{\overset{+}{\text{Cos.}}\,S'H'}.$$

$$\text{or, Cos. } S'H' = \text{Cos. } PS'\,.\,\text{Sec. } PH'.$$

This gives us S′ H and its complement O S′ which is the amplitude.

*To find the Time of Sunrise and Sunset.*

Join Z S′; then, Z S′ being a quadrant, or 90°, we have in triangle Z P S′,

$$\overset{-}{\text{Cos. P}} = -\overset{+}{\text{Cot. P S}'}\ \overset{+}{\text{Cot. P Z}}$$

| | | | | |
|---|---|---|---|---|
| Cot. P S′ | = | 9·923813 | | |
| Cot. P Z | = | 9·561066 | | |
| $\overline{\text{Cos}}$. P | = | 9·484879 | | |
| | | $4^h\ 48^m\ 52^s$ | | |
| | | 12 | | |
| Z P S′ | = | 7 11 8 P.M. | Time of Sunset. | |
| | | 12 | | |
| | | 4 48 52 A.M. | „ | Sunrise. |

The same may be found from the right-angled triangle P S′ H′.

For Cos. P = Tan. P H′ . Cot. P S′.

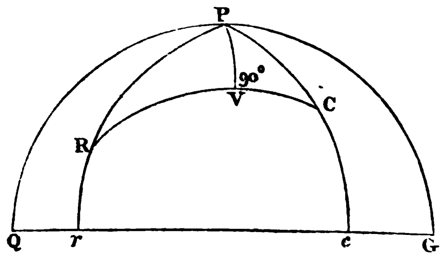
| | | | |
|---|---|---|---|
| | | 9·923813 | |
| | | 9·561066 | |
| | | 9·484879 | |
| H P S′ | = | $4^h\ 48^m\ 52^s$ A.M. | Time of Sunrise. |
| | | 12 | |
| Z P S′ | = | 7 11 8 P.M. | Time of Sunset. |

## GREAT CIRCLE SAILING.

### Example.

To find the distance on the arc of a great circle between Cape Clear and Cape Race; also the latitude and longitude of the vertex.

(1) To find the sides PC, PR, and angle P in triangle RPC.




Lat. Cape-Clear 51° 25′ N. Lat. Cape Race 46° 40′ N.

| | | | |
|---|---|---|---|
| | 90 | | 90 |
| Colat. PC .. | 38 35 | Colat. PR .. | 43 20 |

| | |
|---|---|
| Longitude of Cape Clear .. | 9° 29′ W. |
| „ „ Race .. | 53 3 W. |
| Difference of Long., RPC .. | 43 34 |

(2) To find R C, the distance.

RPC = 43° 34′ Hav. 9·138977

PC = 38 35 Sin. 9·794942
PR = 43 20 Sin. 9·836477

Diff .. 4 45 8·770396

$\theta = 28° 6'$

Vers. 4° 45′ .. 0117873
Vers. 28 6 .. 0003434

0121307 = 28° 31′ (R C.)
60

Distance 1711 miles.

(3) To find Sin. C.

Sin. C : Sin. P :: Sin. PR : Sin. RC.

$$\begin{array}{r} 9{\cdot}838344 \\ 9{\cdot}836477 \\ \hline 19{\cdot}674821 \\ 9{\cdot}678895 \\ \hline \end{array}$$

$\therefore$ Sin. C = 9·995926

Next, draw the arc PV perpendicular to RC, then V will be the Vertex.

(4) To find the Lat. of Vertex.

Sin. PV = Sin. PC, Sin. C.

$$\begin{array}{r} 9{\cdot}794942 \\ 9{\cdot}995926 \\ \hline 9{\cdot}790868 \end{array}$$

$$\begin{array}{lrrr} \text{PV} = & 38^\circ & 9' & 30'' \\ & 90 & & \\ \hline \text{Lat. of Vertex} & 51 & 50 & 30\ \text{N.} \\ \hline \end{array}$$

(5) To find the Long. of Vertex.

Cos. VPC = Tan. PV, Cot. PC.

$$\begin{array}{r} 9{\cdot}895282 \\ 10{\cdot}098099 \\ \hline 9{\cdot}993381 \end{array}$$

$$\begin{array}{lrrrl} \text{Diff. Long. VPC} = & 9^\circ & 58' & 45'' & \text{W.} \\ \text{Long. of Cape Clear} & 9 & 29 & 0 & \text{W.} \\ \hline \text{Long. of Vertex} \ldots & 19 & 27 & 45 & \text{W.} \\ \hline \end{array}$$

*Ans.* Dist. 1,711 miles. Lat. of Vertex 51° 50′ 30″ N. Long. of Vertex 19° 27′ 45″ W.

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NOTE.—When the colatitudes PR, PC are referred to the *South* Pole, the figure should be drawn with P *downwards*, and will therefore be similar to the figure, p. 149, inverted.

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